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Realistic time correlations in sandpiles

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Abstract. – A "sandpile" cellular automaton achieves complex temporal correlations, like a $1/f$ spectrum, if the position where it is perturbed diffuses slowly rather than changing completely at random, showing that the spatial correlations of the driving are deeply related to the intermittent activity. Hence, recent arguments excluding the relevance of self-organized criticality in seismicity and in other contexts are inaccurate. As a toy model of single fault evolution, and despite of its simplicity, our automaton uniquely reproduces the scaling form of the broad distributions of waiting times between earthquakes.

Many complex natural phenomena exhibit quiet periods with a slow loading of the system separated by events in which a rapid evolution takes place [1, 2]. Some examples are earthquakes [3, 4], solar flares [5], creep experiments with cellular glasses [6] and even with piles of rice [7], and rain falls [8]. In these and in other natural and social systems one finds that the intensity of the events and the spatial and temporal scales separating them may vary over broad ranges and often are scale invariant. The similar phenomenologies and the nonequilibrium nature suggest that a common mechanism is emerging in different systems. An interesting candidate is self-organized criticality (SOC) [1, 2, 9], a theory essentially focussing on the occurrence of fast avalanches of nonlinearities, which have been well visualized in simple cellular automata (CA) called "sandpiles".

CA and their probabilistic versions are model systems that have been used quite extensively in the field of complex systems [10, 11]. Good reasons include the possibility of reliable simulations and the conceptual simplicity of the dynamical rules. Furthermore, one has strong indications that the microscopic details of nature are not all-important in deciding certain macroscopic features. Symmetry properties, conservation laws and considerations on the right scale of description matter much more. CA incorporate these essential ingredients and express in the strongest form we know today how emergent behavior can be very rich, varied and complex even if based on few simple dynamical rules on a discrete architecture.

A feature distinguishing several sandpile models from other CA is the scale-free distribution of their avalanche sizes. This resembles the distribution of energy released by earthquakes, for example. On the other hand, temporal correlations between avalanches have been far less investigated, until Boffetta *et al.* noticed that realistic correlations are not found in time series of some sandpiles [12]. Indeed, their avalanches have interoccurrence times distributed exponentially, and the time series appears as a Poisson process of uncorrelated events. This fact was used to argue that SOC cannot be an interpretation of solar-flares activity [12] and, more recently, that SOC is not related to earthquakes [13].

After the critique to SOC by Boffetta *et al.* [12] there has been a wave of investigations on the temporal properties of sandpiles and other SOC models. Nowadays, we know that there are models with interesting temporal correlations, like the $1/f$ decay in power spectra [14–16], or like power-law tails in distributions of waiting times between events [17–22], foreshocks and aftershocks like for earthquakes [23] or even features typical of turbulence [24, 25].

Correlated drivings can give rise to non-exponential waiting-time distributions between events [18], which, *e.g.*, appeared in models of solar activity. In one case the strength of the driving perturbation in the Lu-Hamilton model was slowly varied at random [17]. In another model of "Ellerman bombs" [19], directed percolation was one mechanism increasing the number of driving points. These examples further confirm and illustrate the coexistence of criticality and complex time correlations but it is likely that the correlations intrinsic in these driving mechanisms are simply inherited and show up directly in the output of the automaton.

In this letter we stress that a completely random driving of sandpiles is not expected to be a realistic feature but rather an artificial feature put by hand. In other words, random drivings can be an arbitrary, *a priori* limitation of the automata. Usually sandpiles are driven either at completely random positions or from a fixed single site, but none of these cases reflects the forms of slow loading in real systems, like the crust of the Earth or the solar corona. In seismicity, for example, one observes a rich scaling picture for the waiting-time distributions [3,4], involving location, temporal occurrence, and magnitude of events. In order to have satisfactory CA models of earthquakes, it is therefore desirable to remove excessive randomness from the properties of their driving.

Below, we show how extremely simple spatial correlations in the input drive can generate a complex time series as an output. Thus, the focus is on the *spatial* properties of the nucleating point of the avalanches. As far as we know, we provide the first example of $1/f$ noise in the avalanche activity of a non-running, classical sandpile. Furthermore, we observe broad distributions of waiting times between events also of small sizes. Most importantly, these distributions display a scaling behavior with intensity thresholds, analogous to those recently found for earthquakes [4] and for solar flares [5]. This feature makes at present this new model unique, and corroborates its interpretation in geophysical terms.

The dynamics of the model strongly resembles the one of the sandpile originally discussed by Kadanoff *et al.* [26]: each site i of a linear lattice of side L holds an integer number h_i. In the sandpile jargon, h_i is normally called "number of grains", but this distribution as well may be thought as the potential energy profile in an object mainly extended in one dimension, like a fault. Stable configurations fulfill a local stability condition on the discrete gradient between every pair (i, j) of nearest neighbors,

$$h_i - h_j < H, \tag{1}$$

where the constant H is a threshold. A local instability not fulfilling (1) is resolved by a toppling, which consists in moving α grains from i to j, *i.e.*, $h_i \to h_i - \alpha$ and $h_j \to h_j + \alpha$. In the geophysical interpretation, the toppling thus models a redistribution of energy that takes place after some local elastic threshold within a fault is overtaken.

Stress increase appears as a local increase of potential energy during time step t at position i', causing $h_{i'}(t) = h_{i'}(t-1)+1$. This eventually leads to a violation of (1) between site i' and one or more of its nearest neighbors j. Thus, a toppling takes place between each unstable

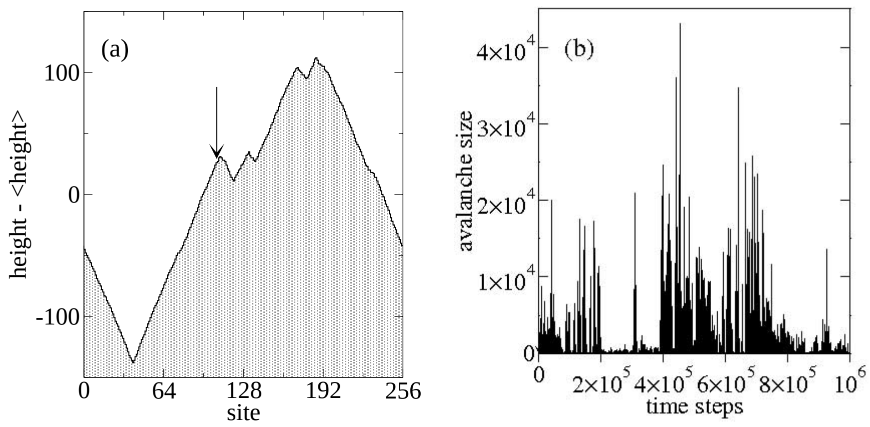

Fig. 1 – (a) Snapshot of a sandpile profile ($L = 256$) around its average height. The arrow points to the position i' where grain addition was taking place. (b) Example of time series of the avalanche sizes for a sandpile with $L = 2048$.

pair (i', j), and j in turn may also become unstable. One then iterates this procedure by making all topplings in parallel as long as some pairs of sites violating (1) are present [27]. The whole set of updates giving rise to a new stable configuration represents an avalanche (earthquake) at time t, with an area a equal to the number of sites that toppled at least once and a size s (energy released) equal to the total number of topplings. As in the standard sandpile scenario, the long time scales of the driving are thus completely separated from the ones of the avalanches. This is the appropriate limit for models of earthquakes, while mixing of time scales is present in solar activity [5], and other models of SOC are possible [22].

A novel feature distinguishing the model proposed here from previous ones is that the site where a grain is added coincides with the position of a random walk: $i'(t)$ with equal probability is drawn from one of the nearest neighbors of $i'(t-1)$. This feature again has a counterpart in seismicity, where it is known that epicenters are spatially clustered and that aftershocks slowly diffuse after large events [28].

The particular driving introduced here makes the model critical also with periodic boundary conditions (BC) [29]. We adopted periodic BC because there cannot be effects related to a fluctuating distance between a predefined boundary and the point where grains are added. For our purposes it is enough to show results on the sandpile in one dimension, with $H = 4$ and $\alpha = 2$. According to our data, the versions with different H or α give similar results. The same remark applies to choices of different mobility; as long as the walker diffuses slowly, no change occurs in the basic aspects of what follows.

A typical sandpile profile is plotted in fig. 1(a). Configurations like that are reached after an appropriate period of fueling of an empty lattice. That period was then not considered in the following statistical analysis. The sandpile profiles observed with periodic BC usually are an alternated sequence of uphills and downhills, thus alternating local minima and maxima. Each patch with the same trend is like a profile of a classical sandpile with open BC, which oscillates around a typical slope. However, grain addition and avalanches make patches to dynamically evolve, merging them or splitting them into shorter pieces.

A time series of the avalanche sizes in the stationary regime is shown in fig. 1(b). It has an intermittent behavior with bursts of activity and temporal clustering of large avalanches, features due to time correlations, as shown below. The probability density of sizes for sev-

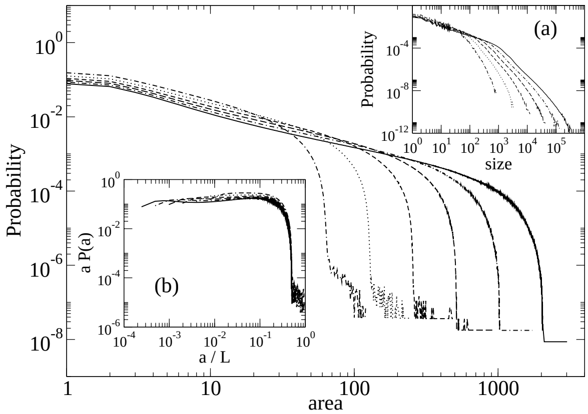

Fig. 2 – Probability density of the area of avalanches, for lattices with $L = 256, 512, \ldots, 4096$, from left to right. The small secondary tail of $P(a)$ comes from rare avalanches with $a > L/2$ involving grains falling onto two sides of a mountain. Insets: (a) size distributions for the same set of L (curves are smoothed). (b) Collapse of $P_L(a)$, according to eq. (2) with $\tau = D = 1$, onto a scaling function F.

eral L's, $P_L(s)$, is shown in inset (a) of fig. 2 and displays multiscaling [30], while the area probability density $P_L(a)$ (fig. 2) obeys simple finite-size scaling

$$P_L(a) \simeq a^{-\tau} F\left(a/L^D\right) \quad (2)$$

as confirmed by a collapse onto a single scaling function $F(x)$ in the plot of $aP_L(a)$ *vs.* a/L (see inset (b) of fig. 2). The model thus has a critical behavior, with $\tau \simeq D \simeq 1$ [31].

Complex temporal correlations and long memory are revealed by the non-trivial form of the power spectrum $S(f)$ of the time series of avalanche sizes $s(t)$ (with $s(t) = 0$ if at time t there were no topplings). Figure 3(a) shows that at quite low frequencies $S(f) \sim 1/f^\gamma$ with $\gamma = 1.00(1)$. In fig. 3(b) we plot $S(f)$ *vs.* fL to show that the $1/f$ region becomes broader when L in increased: it is bounded on the right by a bump at $f \sim 1/L$, while the crossover frequency to a flat spectrum at very low f scales approximately as $1/L^2$. Almost equal spectra are observed for the time series of avalanche areas, confirming that the basic origin of that $1/f$ spectrum is in the temporal correlations of events rather than in their actual intensities. We remind that a $1/f$ spectrum is observed in many natural phenomena [32], and it is regarded as one of the most complex forms of time correlation. It is remarkable that the long-range memory in the model arises by the non-trivial, self-organized structure encoded in the sandpile profile, in agreement with a basic idea of SOC theory, namely that non-equilibrium extended systems can self-organize to complex regimes even if their dynamical rules are simple.

The other form of temporal correlation concerns the waiting-time distribution between events which has a power law tail, rather than an exponential decay as for uncorrelated events. The waiting-time distributions $P_s(t_w)$ between avalanches of size $\geq s$ in a sandpile with $L = 4096$ are plotted in fig. 4(a) for some thresholds s ranging from $2^8 = 64$ to $2^{17} = 131072$. Already for a small fixed s, the distributions have a power law tail, which does not disappear when larger L are considered. In the inset of fig. 4(a), for example, one can see that $P_{s=64}(t_w)$ has the same power law tail for $L = 256$ and $L = 4096$. Interestingly, in fact when $s \gtrsim L/2$ (steepest part of $P_L(s)$, see inset (a) of fig. 2) in $P_s(t_w)$ there are two regimes with two different power law decays, as one observes in the analysis of waiting times of earthquakes [4] and for solar flares [5]. As in these cases, we find it appropriate to rescale times by multiplying them by rates of events $\geq s$, denoted as $R(s)$ [33].

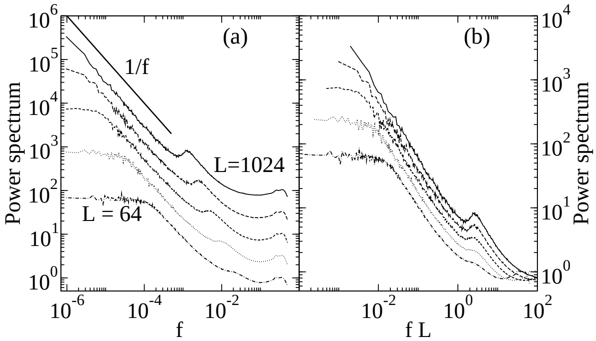

Fig. 3 – Log-log plot of the power spectrum of the time series of avalanche sizes, for $L = 64$, 128, 256, 512, and 1024, (a) *vs.* the frequency and (b) *vs.* fL. In (a) curves have been offset half-decade from each other, and a straight line indicates a $1/f$ scaling. For the estimates of $S(f)$, we used time windows with 2^{21} data.

The rescaled times $T = t_w R(s)$ appear to be the natural ones, because $P_s(t_w)/R(s)$ *vs.* $t_w R(s)$ collapse quite well onto a single scaling curve $G(T)$ (see fig. 4(b)). Since $G(T)\mathrm{d}T = P_s(t_w)\mathrm{d}t_w$, we see that $G(T)$ is the probability density of T, not depending anymore on thresholds. Hence, a unifying scheme can be applied to all thresholds also in our sandpile model, showing that there is a basic self-similar mechanism taking place. For short rescaled times $G(T) \sim T^{-\beta_1}$ with $\beta_1 = 0.8(1)$, while $G(T) \sim T^{-\beta_2}$ with $\beta_2 = 2.0(2)$ for $T \gg 1$. (For earthquakes Corral found $\beta_1 = 0.9(1)$ and $\beta_2 = 2.2(1)$ [4].) The crossover between the two power law regimes is around $T^* \approx 0.7$. Thus, for t_w much smaller than the average occurrence time $\bar{t}(s) = R(s)^{-1}$ of events with size $\geq s$ (*i.e.* $T \ll 1$), the correlations between events are qualitatively different from the ones at $t_w \gg \bar{t}(s)$.

A mechanism giving rise to strong temporal correlations between avalanches should be their spatial overlap, since the memory of past activity is stored in the height profile of the sandpile. By dropping grains at a point slowly diffusing, as we do, a fresh part of the profile is

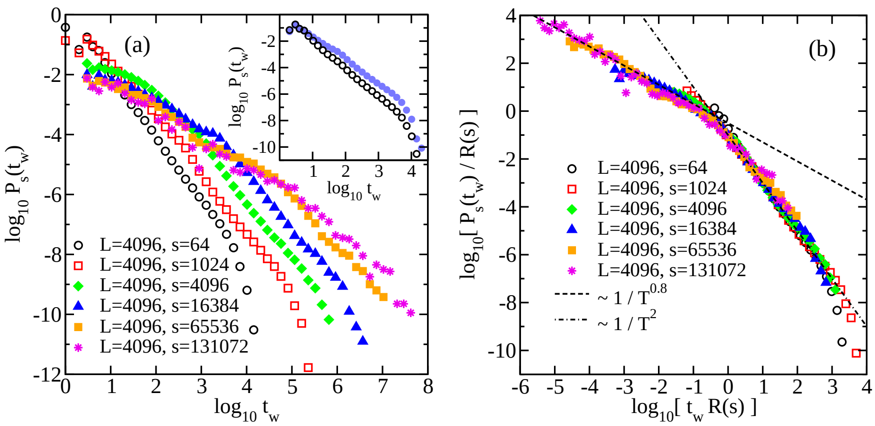

Fig. 4 – (a) Log-log plot of distributions of waiting times between events with size $\geq s$, for $L = 4096$ and various s. Inset: the cases ($L = 256$, $s = 64$; $\bullet$) and ($L = 4096$, $s = 64$; $\circ$) are compared. (b) Log-log plot of the same distributions, but rescaled with the rate of events $\geq s$.

almost always found, probably enhancing the correlation with past activity. In some classical models driven at random points, scale-free waiting-time distributions are observed between sufficiently large avalanches [18], which indeed have a consistent probability to overlap with previous ones. On the other hand, that argument does not really explain why the $1/f$ part of the spectrum of our sandpile extends to very low frequencies.

We have mainly discussed SOC in CA, but a different class of SOC models also supports the view that space and time correlations are related to each other. We are referring to models with "extremal dynamics". In these models each unit carries a continuum variable and the unit closer to instability is always the one relaxing (toppling). The Olami-Feder-Christensen (OFC) model of earthquakes [34] is representative of extremal dynamics. Whether criticality in the OFC model is only apparent or persists in the limit $L \to \infty$ is an open issue. Nevertheless, the OFC model has complex patterns in time [20, 23] and space: indeed, by its nature, the OFC map self-organizes also the position of the epicenters (driving points), which form sequences deeply linked to past activity [35]. Similarly, in the Bak-Sneppen model of evolution, distances between extremal sites follow a Levy flight, and $1/f^\gamma$ spectra are found with $0 < \gamma < 1$ [36]. Recently, Lippiello *et al.* introduced a hybrid sandpile/extremal dynamics model of seismicity that also displays waiting-time distributions with power law tails [21].

In summary, we have studied a sandpile cellular automaton exhibiting self-organized criticality (even without open boundaries). A critical stationary state with correlated avalanches and intermittent transport takes place because the model is perturbed at a position slowly diffusing in space, a process leading to clustered nucleation points and motivated by the observation of natural phenomena like earthquakes, where epicenters are correlated. Indeed, the distributions of waiting times between avalanches larger than given thresholds have striking and unique similarities with the ones found in real seismicity. We can thus reproduce some of the complexity of natural critical phenomena, involving jointly energetic, spatial and temporal aspects. Our results suggest that the logic that led to the present model is right, namely that spatial correlations should not be arbitrarily removed by imposing a random drive in models of critical phenomena like sandpiles, which are by themselves already very simplified objects. The positive comparison of our results with earthquake activity leads to propose that a rugged landscape as depicted in fig. 1(a) may be related to the stress distribution along a fault. The absence of a single slope profile (as observed in sandpiles with open boundaries) implies that system-wide events are not always possible, with potential interesting implications about predictability of large events.

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