Extended Supersymmetry in Massless Conformal Higher Spin Theory
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1 Introduction

2 Superparticle in $\mathcal{N}$-extended tensorial superspace

3 Conformal higher spin equations in $\Sigma\left(\frac{n(n+1)}{2}\right|\mathcal{N}n)$ with even $\mathcal{N}$

4 From the superfield to the component form of the higher spin equations in tensorial space

5 Conclusions and Discussion
String Theory is believed to be a selfconsistent finite quantum theory of gravity and all fundamental interactions.
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Understanding of the dynamics of higher spin states is important for a deeper comprehension of String Theory.
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Also the problem of constructing an interacting theory of HSF is one of the fundamental problems of the theoretical physics
A theory of interacting massless higher spin fields (MHSF) in $AdS_4$ was constructed by Vasiliev [’90].
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This is why it is of interest to study possible reformulations of Vasiliev theory/different formulations of HS theory, beginning from the case of FHS theory
Introduction

D=4 Free conformal higher spin → Field theory on tensorial space \( \Sigma(10|0) \)

\[
X_{\alpha\beta} = X_{\beta\alpha} = \frac{1}{4} x_{\gamma} x_{\alpha\beta} + \frac{1}{8} y_{mn} \gamma_{\alpha\beta mn} = 1, \ldots, 4, m, n = 0, \ldots, 3
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Fronsdal was discussed this space as a basis for constructing MCHS in 1986.

First dynamical system was actually formulated in tensorial SSP \( \Sigma(10|4) \)

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Z^M := (X_{\alpha\beta}, \theta^\alpha)
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This was generalized superparticle model [Bandos, Lukierski '98]

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The superparticle model [Bandos, Lukierski ’98] was actually formulated in more general superspace

\[\Sigma \left( \frac{n(n+1)}{2} \right) \mid n \right): \quad Z^M := (X^{\alpha\beta}, \theta^\alpha), \quad \begin{cases} \alpha, \beta = 1, \ldots, n, \\ X^{\alpha\beta} = X^{\beta\alpha} \end{cases} \]
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\[ \Sigma \left( \frac{n(n+1)}{2} | n \right): \quad Z^M := (X^\alpha \beta, \theta^\alpha), \] \[ \begin{align*}
\alpha, \beta &= 1, \ldots, n, \\
X^\alpha \beta &= X^\beta \alpha
\end{align*} \]

Taking in mind the spinorial treatment of \( \alpha, \beta \) indices \( \rightarrow \) 
\[ n = 2^k = 2, 4, 8, 16, \ldots \]
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Taking in mind the spinorial treatment of \(\alpha, \beta\) indices \(\rightarrow\)

\(n = 2^k = 2, 4, 8, 16, \ldots\)

\(n=4,8,16\) tensorial SSP can be used to describe \(D = 4, 6, 10\) FMCHS theories [Bandos, Lukierski, Sorokin ’99], [Bandos, Bekaert, de Azcarraga, Sorokin, Tsulaia JHEP05]
The $\mathcal{N} = 1$ supermultiplets of $D=4,6,10$ CHS fields are described by the real scalar superfields on the $n=4,8,16 \sum \left( \frac{n(n+1)}{2} \right) |n|$ superspaces

$$\Phi(X^{\alpha\beta}, \theta^\gamma) = b(X) + f_{\alpha}(X) \theta^\alpha + \sum_{i=2}^{n} \phi_{\alpha_1 \ldots \alpha_i}(X) \theta^{\alpha_1} \ldots \theta^{\alpha_i}$$
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- Obeying
  - $D_{[\alpha} D_{\beta]} \Phi(X, \theta) = 0$
  - $D_\alpha = \frac{\partial}{\partial X^\alpha} + i \theta^\beta \partial_{\beta\alpha}$
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\[ \phi_{\alpha_1...\alpha_i}(X) = 0, \quad i > 1; \]

$\to b(x)$ and $f_\gamma(x)$, obey the eqs. [Vasiliev’01]

\[ \partial_{\alpha[\beta} \partial_{\gamma]} \delta b(X) = 0, \quad \partial_{\alpha[\beta} f_\gamma](x) = 0 \]
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We fill this gap by presenting the free $\mathcal{N}=2,4,8$ supersymmetric CHS equations in $\mathcal{N}$–extended tensorial superspaces $\Sigma^{(\frac{n(n+1)}{2})|\mathcal{N} n}$.
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One of the reasons of our interest in $\mathcal{N}$-extended SUSY systems in tensorial SSP comes from the observation that $\mathcal{N}$-extended SUSY with $\mathcal{N} = 4$ unifies the scalar and vector gauge fields.
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One of the reasons of our interest in $\mathcal{N}$-extended SUSY systems in tensorial SSP comes from the observation that $\mathcal{N}$-extended SUSY with $\mathcal{N} = 4$ unifies the scalar and vector gauge fields.

The study of $\mathcal{N}$-extended supersymmetries might be convenient in a search for a sensible generalization of Maxwell and Einstein equations in tensorial space (presently, only generalizations of K-G eqs and D eqs are known).
An action for the $\Sigma^{(\frac{n(n+1)}{2})\mid\mathcal{N}}$ superparticle

$$S = \int d\tau \left[ \dot{X}^{\alpha\beta}(\tau) - i\dot{\theta}^{\alpha I}(\tau)\dot{\bar{\theta}}^{\beta I}(\tau) \right] \lambda_{\alpha}(\tau)\lambda_{\beta}(\tau), \begin{cases} \alpha, \beta = 1, \ldots, n, \\ I = 1, 2, \ldots, \mathcal{N} \end{cases}$$
An action for the $\Sigma(\frac{n(n+1)}{2}|N\ n)$ superparticle

$$S = \int d\tau \left[ \hat{X}^{\alpha\beta}(\tau) - i\hat{\theta}^{\alpha I}(\tau)\hat{\theta}^{\beta I}(\tau) \right] \lambda_\alpha(\tau)\lambda_\beta(\tau), \quad \left\{ \begin{array}{l} \alpha, \beta = 1, \ldots, n, \\ I = 1, 2, \ldots, N \end{array} \right.$$ 

The action is manifestly invariant under the supertranslations of the rigid $N$-extended tensorial superspace $\Sigma(\frac{n(n+1)}{2}|Nn)$

$$\delta_a X^{\alpha\beta} = a^{\alpha\beta}, \quad \delta_a \theta^{I\alpha} = 0, \quad \delta_\epsilon X^{\alpha\beta} = i\theta^{I(\alpha} \epsilon^{\beta)I}, \quad \delta_\epsilon \theta^{I\alpha} = \epsilon^{\beta I}$$
An action for the $\Sigma^{(n(n+1)/2)Nn}$ superparticle

$$S = \int d\tau [\dot{X}^{\alpha\beta}(\tau) - i\dot{\theta}^{\alpha l}(\tau)\dot{\theta}^{\beta l}(\tau)]\lambda_{\alpha}(\tau)\lambda_{\beta}(\tau), \begin{cases} \alpha, \beta = 1, \ldots, n, \\ l = 1, 2, \ldots, N \end{cases}$$

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Which act on the worldline fields as

$$\delta_a \dot{X}^{\alpha\beta}(\tau) = a^{\alpha\beta}, \quad \delta_a \dot{\theta}^{l\alpha} = 0; \quad \delta_a \dot{\lambda}_\alpha = 0,$$

$$\delta_\epsilon \dot{X}^{\alpha\beta} = i\dot{\theta}^{l(\alpha} \epsilon^{\beta)l}, \quad \delta_\epsilon \dot{\theta}^{l\alpha} = \epsilon^{\beta l}; \quad \delta_\epsilon \dot{\lambda}_\alpha = 0.$$
Symplectic supertwistor form of the action

Using Leibniz's rule we can rewrite the action in the form

\[ S = \int W^1 (\lambda_\alpha d\mu^\alpha - \mu^\alpha d\lambda_\alpha - i d\chi^I \chi^I) \]
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\[ S = \int_{W^1} (\lambda_\alpha d\mu^\alpha - \mu^\alpha d\lambda_\alpha - i d\chi^I \chi^I) \]

The relations between new bosonic spinor \( \mu^\alpha \), fermionic scalar \( \chi^I \) and the variables of the original action are

\[ \mu^\alpha = \hat{X}^{\alpha\beta} \lambda_\beta - \frac{i}{2} \hat{\theta}^{\alphaI} \chi^I, \quad \chi^I = \hat{\theta}^{\alphaI} \lambda_\alpha \]

This generalizes the Penrose-Feber-Shirafuji incidence relations.
Symplectic supertwistor form of the action

\[ S = \int_{W^1} \left( \lambda_\alpha d\mu^\alpha - \mu^\alpha d\lambda_\alpha - i d\chi^I \chi^I \right) = \int_{W^1} d\Upsilon^\Sigma \Xi^{\Sigma \Omega} \Upsilon^\Omega \]
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Orthosymplectic supertwistor (=fundamental reps of $OSp(1|2n)$)

$$\gamma^\Sigma = \begin{pmatrix} \mu^\alpha \\ \lambda_\alpha \\ \chi^I \end{pmatrix}, \Xi^{\Sigma \Omega} = \begin{pmatrix} 0 & \delta_\alpha^\beta & 0 \\ -\delta^\alpha_\beta & 0 & 0 \\ 0 & 0 & -i\delta^I_J \end{pmatrix}, \begin{cases} \alpha, \beta = 1, \ldots, n \\ I = 1, \ldots, \mathcal{N} \end{cases}$$
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Since this action does not possess any gauge symmetries, the components of the orthosymplectic supertwistor are the true, physical degrees of freedom ($2n$ bosonic and $N$ fermionic) of our generalized superparticle model.
The first action depends on more bosonic and fermionic variables. Hence, it should possess bosonic and fermionic GS which reduce the number of physical degrees of freedom to those in the supertwistor $\gamma^\Sigma$ in the action.
Gauge symmetries (GS)

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Gauge symmetries: Fermionic $\delta_\kappa$ and bosonic $\delta_b$

\[
\delta_\kappa \hat{X}^{\alpha\beta} = i \delta_\kappa \hat{\theta}^l(\alpha \hat{\theta}^\beta)^l, \quad \delta_\kappa \hat{\theta}^l \lambda_\alpha = 0, \quad \delta_\kappa \lambda_\alpha = 0,
\]

\[
\delta_b \hat{X}^{\alpha\beta} \lambda_\alpha = 0, \quad \delta_b \hat{\theta}^l \lambda_\alpha = 0, \quad \delta_b \lambda_\alpha = 0.
\]
Constraints and their conversion to first class

The \( \delta_\kappa \) and \( \delta_b \) GS are generated by 1st class constraints which may be extracted from the bosonic and fermionic primary constraints

\[
d_{\alpha l} := \pi_{\alpha l} + iP_{\alpha\beta}\theta^{\beta l} \approx 0 \, , \quad P_{\alpha\beta} := P_{\alpha\beta} - \lambda_\alpha\lambda_\beta \approx 0 \, , \quad P^{\alpha(\lambda)} \approx 0
\]

Where:

\[
P_{\alpha\beta} := \frac{1}{2} \frac{\delta L}{\delta \dot{X}_{\alpha\beta}} \, , \quad P^{(\lambda)}_{\alpha} := \frac{\delta L}{\delta \dot{\lambda}_{\alpha}} \, , \quad \pi_{\alpha l} := \frac{\delta L}{\delta \dot{\theta}_{\alpha l}}
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Where:

$$P_{\alpha \beta} := \frac{1}{2} \frac{\delta L}{\delta \dot{X}^{\alpha \beta}}, \quad P^{(\lambda)}_{\alpha} := \frac{\delta L}{\delta \dot{\lambda}_\alpha}, \quad \pi_{\alpha l} := \frac{\delta L}{\delta \dot{\theta}_{\alpha l}}$$

The nonvanishing Poisson brackets of the above constraints are

$$\{d_{\alpha l}, d_{\beta J}\}_{PB} = -2iP_{\alpha \beta} \delta_{IJ}, \quad [P_{\alpha \beta}, P^{\gamma (\lambda)}]_{PB} = -2\lambda_{(\alpha \delta \beta)^\gamma}$$
Conversion procedure

Add a pair of degrees of freedom to each pair of second class constraints to modify them in such a way that they form a closed algebra.

The bosonic sector is the same irrespective of $N$ and the conversion is there effectively reduced to ignoring $P_\alpha(\lambda) \approx 0$.

$N = 1$ [Bandos, Lukierski, Sorokin '99]
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Add a pair of degrees of freedom to each pair of second class constraints to modify them in such a way that they form a closed algebra.

The bosonic sector is the same irrespective of $\mathcal{N}$ and the conversion is there effectively reduced to ignoring $P^{\alpha(\lambda)} \approx 0$ for $\mathcal{N} = 1$ [Bandos, Lukierski, Sorokin ’99].
For the fermionic sector we introduce $\mathcal{N}$ fermionic variables $\chi^I$ and we postulate a Clifford-type PB for them

$$\{\chi^I, \chi^J\}_{PB} = -2i\delta^{IJ}$$
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After conversion, the superparticle model is described by the following set of 1st class constraints

$$\mathcal{D}_{\alpha I} := \pi_{\alpha I} + iP_{\alpha \beta} \theta^\beta I + i\chi^I \lambda_\alpha \approx 0$$

$$\mathcal{P}_{\alpha \beta} := P_{\alpha \beta} - \lambda_\alpha \lambda_\beta \approx 0$$

Obeying the following superalgebra:

$$\{\mathcal{D}_{\alpha I}, \mathcal{D}_{\beta J}\}_{PB} = -2i\mathcal{P}_{\alpha \beta} \delta^{IJ}$$

$$[\mathcal{P}_{\alpha \beta}, \mathcal{D}_{\gamma I}]_{PB} = 0$$

$$[\mathcal{P}_{\alpha \beta}, \mathcal{P}_{\gamma \delta}]_{PB} = 0$$
Quantization of the superparticle in $\sum (\frac{n(n+1)}{2}|N_n)$ and conformal higher spin superfield equations

Quantization à la Dirac

Impose the quantum 1st class constraints as equations to be satisfied by its wavefunction (WF)
Quantization of the superparticle in $\Sigma(\frac{n(n+1)}{2} | N n)$ and conformal higher spin superfield equations

Quantization à la Dirac

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WF depends on $X^{\alpha\beta}$, $\theta^{\alpha l}$, $\lambda_\alpha$ and a half of the $\chi^l$

$$\eta^q = \frac{\chi^q - i\chi^N/2 + q}{2}, \quad \bar{\eta}_q = \frac{\chi^q + i\chi^N/2 + q}{2}, \quad \{\bar{\eta}_q, \eta^p\}_{PB} = -i\delta_{qp}$$
Quantization of the superparticle in $\sum\left(\frac{n(n+1)}{2}\right)|N^n|$ and conformal higher spin superfield equations

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$\bar{\eta}_q$ can be treated as momentum for $\eta_q$ and the WF depends only on $\eta^q$

$$\mathcal{W} = \mathcal{W}(X^{\alpha\beta}, \theta^\alpha; \lambda_\alpha; \eta^q)$$
Quantization of the superparticle in $\sum \frac{n(n+1)}{2} |Nn\rangle$ and conformal higher spin superfield equations
Quantization of the superparticle in $\sum \left( \frac{n(n+1)}{2} |Nn\right)$ and conformal higher spin superfield equations

Decomposing the WF in a finite power series in $\eta^q$ and imposing the fermionic constraints, we obtain

$$W(X, \Theta^q, \bar{\Theta}^q, \lambda, \eta^q) = W(0) + \frac{N}{2} \sum_{k=1}^{1} k! \eta^q \cdots \eta^1 W(k) q_1 \cdots q_k$$

$$D_\alpha q W(0) = 2 \lambda_\alpha W(1) q_1, \ldots, D_\alpha q W(k) q_1 \cdots q_k = 2 \lambda_\alpha W(k+1) q_1 q_2 \cdots q_k$$

$$\bar{D}_\alpha q W(N/2) q_1 \cdots q_{N/2} = N \lambda_\alpha W(N/2 - 1) q_1 \cdots q_{N/2 - 1} \delta q_{N/2}$$

$$w(\lambda, \Theta^q \lambda) = w(0) (\lambda) + \frac{N}{2} \sum_{k=1}^{1} k! (\lambda \Theta^q k) \cdots (\lambda \Theta^q 1) w(k) q_1 \cdots q_k (\lambda)$$

Carlos Meliveo (UPV-EHU)
Quantization of the superparticle in $\sum \left( \frac{n(n+1)}{2} |Nn\right)$ and conformal higher spin superfield equations

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- $\bar{\mathcal{D}}_{\alpha q} W^{(0)} = 0$, $\bar{\mathcal{D}}_{\alpha q} W^{(k)}_{q_1 \ldots q_k} = 2k\lambda_{\alpha} W^{(k-1)}_{[q_1 \ldots q_{k-1}] \delta_{q_k} q}$
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- $\bar{D}^q_{\alpha} W^{(N/2)}_{q_1 \ldots q_{N/2}} = N\lambda_\alpha W^{(N/2-1)}_{[q_1 \ldots q_{N/2-1}} \delta^{q_{N/2}]}$
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\[
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\]

\[
D_{\alpha q} \mathcal{W}^{(0)} = 2\lambda_{\alpha} \mathcal{W}_q^{(1)}, \ldots, D_{\alpha q} \mathcal{W}_{q_1 \ldots q_k}^{(k)} = 2\lambda_{\alpha} \mathcal{W}_{qq_1 \ldots q_k}^{(k+1)}
\]

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\bar{D}^q_{\alpha} \mathcal{W}^{(0)} = 0, \bar{D}^q_{\alpha} \mathcal{W}_{q_1 \ldots q_k}^{(k)} = 2k\lambda_{\alpha} \mathcal{W}_{[q_1 \ldots q_{k-1} \delta_{q_k}]q}^{(k-1)}
\]

\[
\bar{D}^q_{\alpha} \mathcal{W}_{q_1 \ldots q_{N/2}}^{(N/2)} = N\lambda_{\alpha} \mathcal{W}_{[q_1 \ldots q_{N/2-1} \delta_{q_{N/2}}]q}^{(N/2-1)}
\]

\[
\mathcal{W}^{(0)}(X, \Theta^q, \bar{\Theta}_q, \lambda) = w(\lambda, \Theta^q, \lambda) \exp[i\lambda_{\alpha} \lambda_{\beta}(X + 2[\Theta^p \bar{\Theta}_p]^{q(p)})].
\]
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- $w(\lambda, \Theta^q \lambda) = w^{(0)}(\lambda) + \sum_{k=1}^{N/2} \frac{1}{k!} (\lambda \Theta^{q_k}) \ldots (\lambda \Theta^{q_1}) W^{(k)}_{q_1 \ldots q_k}(\lambda)$
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\]

- \( D_{\alpha q} W^{(0)} = 2\lambda_{\alpha} W_q^{(1)} \), \( D_{\alpha q} W_{q_1 \cdots q_k}^{(k)} = 2\lambda_{\alpha} W_{qq_1 \cdots q_k}^{(k+1)} \)

- \( \bar{D}_{\alpha} W^{(0)} = 0 \), \( \bar{D}_{\alpha} W_{q_1 \cdots q_k}^{(k)} = 2k\lambda_{\alpha} W_{[q_1 \cdots q_{k-1}} \delta_{q_k]}^{(k-1)} \)

- \( \bar{D}_{\alpha} W_{q_1 \cdots q_{N/2}}^{(N/2)} = N\lambda_{\alpha} W_{[q_1 \cdots q_{N/2-1}} \delta_{q_{N/2]}^{(N/2-1)} \}

- \( W^{(0)}(X, \Theta^q, \bar{\Theta}_q, \lambda) = w(\lambda, \Theta^q \lambda) \exp\{i\lambda_{\alpha} \lambda_{\beta}(X + 2i \Theta^p \bar{\Theta}_p)^{\alpha\beta}\} \)

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Quantization of the superparticle in $\sum \binom{n(n+1)}{2} |N_n|$ and conformal higher spin superfield equations

To obtain a WF in tensorial coordinate representation one has to integrate $w(\lambda, \Theta^q \lambda)$ over $\lambda_\alpha$

$$\Phi(X, \Theta^q, \bar{\Theta}_q) = \int d^n \lambda w(\lambda, \Theta^q \lambda) e^{i\lambda_\alpha \lambda_\beta (X + 2i \Theta^p \bar{\Theta}_p)^{\alpha\beta}}$$
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The superfield $\Phi$ is chiral $\bar{D}^q_{\alpha} \Phi(X, \Theta^q', \bar{\Theta}_{p'}) = 0$ and satisfies the equation $D_q[\beta D_{\gamma}p] \Phi(X, \Theta^q', \bar{\Theta}_{p'}) = 0$
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- These superfield eqs. $\bar{D}^q_\alpha \Phi(X, \Theta^q, \bar{\Theta}_{p'}) = 0$, $D_q[\beta D_{\gamma}]_p \Phi(X, \Theta^q, \bar{\Theta}_{p'}) = 0$ are our main result.
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- We show that they describe supermultiplets of FMCHS fields in $N$–extended tensorial superspace with even $N$
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$$

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- We show that they describe supermultiplets of FMCHS fields in $\mathcal{N}$–extended tensorial superspace with even $\mathcal{N}$
$\mathcal{N} = 2$ Supersymmetric FMCHS theory

All the components of the 'chiral' superfield vanish except the first two which obey the FHSE

$$\Phi(X, \Theta, \bar{\Theta}) = \phi(X_L) + i \Theta^\alpha \psi_\alpha(X_L), \quad X_L^{\alpha\beta} = X^{\alpha\beta} + 2i \Theta^{(\alpha} \bar{\Theta}^{\beta)},$$
\( N = 2 \) Supersymmetric FMCHS theory

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\]

With \( \phi \) and \( \psi_\alpha \) obeying the Vasiliev's FHSEs

\[
\partial_{\alpha[\gamma} \partial_{\delta]} \phi(X) = 0, \quad \partial_{\alpha[\beta} \psi_{\gamma]}(X) = 0
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\partial_{\alpha[\gamma} \partial_{\delta]} \phi(X) = 0, \quad \partial_{\alpha[\beta} \psi_{\gamma]}(X) = 0
\]

The \( \mathcal{N} = 2 \) supermultiplet is given by a complexification of the \( \mathcal{N} = 1 \) supermultiplet
When $\mathcal{N} > 2$ spin-tensorial components are present. The general solution for $\mathcal{N} = 4$

\[
\Phi(X, \Theta^q, \bar{\Theta}_q) = \phi(X_L) + i\Theta^\alpha q \psi_{\alpha q}(X_L) + \epsilon_{pq} \Theta^\alpha q \Theta^\beta p F_{\alpha \beta}(X_L),
\]

\[
X^{\alpha \beta}_L = X^{\alpha \beta} + 2i\Theta^q(\alpha \bar{\Theta}^\beta_q), \quad q = 1, 2
\]

where $\phi$ and $\psi_{\alpha q}$ obey the Vasiliev’s HSEs

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\partial_{\alpha [\gamma} \partial_{\delta] \beta} \phi(X) = 0, \quad \partial_{\alpha [\beta} \psi_{\gamma] q}(X) = 0
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$$X^\alpha_\beta = X^{\alpha \beta} + 2i \Theta^q (\alpha \bar{\Theta}^\beta)_q, \quad q = 1, 2$$

where $\phi$ and $\psi_\alpha q$ obey the Vasiliev’s HSEs

$$\partial_{\alpha [\gamma} \partial_{\delta]} \phi(X) = 0, \quad \partial_{\alpha [\beta} \psi_{\gamma]} q(X) = 0$$

While the complex symmetric bi-spinor $\mathcal{F}_{\alpha \beta} = \mathcal{F}_{\beta \alpha}$ satisfies

$$\partial_{\alpha [\gamma} \mathcal{F}_{\delta]} \beta(X) = 0, \quad \mathcal{F}_{\alpha \beta} = \mathcal{F}_{\beta \alpha}$$
\( \mathcal{N} = 4 \) Supersymmetric FMCHS theory

This looks like a tensorial space counterpart of the \( D = 4 \) Maxwell equations
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But its general solution reads

$$\mathcal{F} = \partial_{\alpha\beta} \tilde{\phi}, \quad \text{where} \quad \partial_{\alpha[\beta} \partial_{\gamma]} \delta \tilde{\phi} = 0$$
$\mathcal{N} = 4$ Supersymmetric FMCHS theory

This looks like a tensorial space counterpart of the $D = 4$ Maxwell equations

But its general solution reads

$$\mathcal{F} = \partial_{\alpha\beta} \tilde{\phi}, \quad \text{where} \quad \partial_{[\alpha} \partial_{\beta] \delta} \tilde{\phi} = 0$$

Thus the $\mathcal{N} = 4$ HS supermultiplet contains two complex scalar fields and two Dirac spinor fields, $\phi(X), \psi_1^\alpha(X), \psi_2^\alpha(X), \tilde{\phi}(X)$
They appear in the on-shell scalar superfield decomposition as

\[
\Phi(X, \Theta^q, \bar{\Theta}_q) = \phi(X_L) + i\Theta^a \psi_\alpha(X_L) + \epsilon_{pq} \Theta^a \Theta^\beta \partial_{\alpha \beta} \tilde{\phi}(X_L)
\]
\( \mathcal{N} = 4 \)

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\]

With

\[
\partial_{\alpha [\gamma} \partial_{\delta]} \phi(X) = 0, \quad \partial_{\alpha [\beta} \psi_{\gamma]} q(X) = 0, \quad \partial_{\alpha [\beta} \partial_{\gamma]} \partial_{\delta} \tilde{\phi} = 0
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\( N = 4 \)

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\]

With

\[
\partial_{\alpha[\gamma} \partial_{\delta]} \phi(X) = 0, \quad \partial_{\alpha[\beta} \psi_{\gamma]} q(X) = 0, \quad \partial_{\alpha[\beta} \partial_{\gamma]} \tilde{\phi} = 0
\]

The 2nd complex scalar field \( \tilde{\phi} \) enters the original superfield with a derivative \( \rightarrow \) the model possesses the Peccei-Quinn-like symmetry (PQS)

\[
\tilde{\phi}(X) \mapsto \tilde{\phi}(X) + \text{const}
\]
For $N = 8$ our equations $\bar{D}_\alpha^q \Phi = 0 = D_q[\alpha D_\beta]_\rho \Phi$ are solved by the following SF

$$\Phi(X, \Theta^q, \bar{\Theta}_q) = \phi(X_L) + i \Theta^{\alpha q} \psi_{\alpha q}(X_L) + \frac{1}{2} \Theta^{\alpha_2 q_2} \Theta^{\alpha_1 q_1} F_{\alpha_1 \alpha_2 \ q_1 q_2}(X_L) +$$

$$+ \frac{i}{3!} \Theta^{\alpha_3 q_3} \Theta^{\alpha_2 q_2} \Theta^{\alpha_1 q_1} \epsilon_{q_1 q_2 q_3 q} \psi^{q}_{\alpha_1 \alpha_2 \alpha_3}(X_L) +$$

$$+ \frac{1}{4!} \epsilon_{q_1 q_2 q_3 q_4} \Theta^{\alpha_4 q_4} \ldots \Theta^{\alpha_1 q_1} F_{\alpha_1 \ldots \alpha_4}(X_L)$$

where $\phi, \psi_{\alpha q}$ obey the Vasiliev’s HSEs and

$$\partial_{\alpha [\gamma} F_{\delta \beta] q_1 q_2}(X) = 0 \rightarrow F_{\alpha \beta \ q_1 q_2}(X) = \partial_{\alpha \beta} \phi_{q_1 q_2}(X),$$

$$\partial_{\alpha [\gamma} \psi_{\delta \beta_2 \beta_3 q}(X) = 0 \rightarrow \psi^{q}_{\alpha_1 \alpha_2 \alpha_3}(X) = \partial_{\alpha_1 \alpha_2} \tilde{\psi}^{q}_{\alpha_3}(X),$$

$$\partial_{\alpha [\gamma} F_{\delta \beta_2 \beta_3 \beta_4}(X) = 0 \rightarrow F_{\alpha \ldots \alpha_4}(X) = \partial_{\alpha_1 \alpha_2} \partial_{\alpha_3 \alpha_4} \tilde{\phi}(X)$$
Hence the $\mathcal{N} = 8$ supermultiplet of FHSF is described in tensorial space by a set of two scalar fields, a sextuplet of scalar fields, a spinor field and a quadruplet of spinorial fields obeying

$$
\partial_{\alpha}[\gamma\partial_{\delta}\beta]\phi(X) = 0, \quad \partial_{\alpha}[\beta\psi_{\gamma}](X) = 0, \quad \partial_{\alpha}[\gamma\partial_{\delta}\beta]\phi_{qp}(X) = 0, \\
\partial_{\alpha}[\beta\tilde{\psi}_{\gamma}]^q(X) = 0, \quad \partial_{\alpha}[\gamma\partial_{\delta}]\beta\tilde{\phi}(X) = 0
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\partial_{\alpha[\beta} \tilde{\psi}_{\gamma]}^q(X) = 0, \quad \partial_{\alpha[\gamma} \partial_{\delta}\beta \tilde{\phi}(X) = 0
\]

The ’higher’ scalar and spinor fields enter the original superfield with one or two derivatives

\[
\phi_{qp}(X) \mapsto \phi_{qp}(X) + a_{qp}, \\
\tilde{\psi}_{\alpha}^q(X) \mapsto \tilde{\psi}_{\alpha}^q(X) + \beta_{\alpha}^q, \\
\tilde{\phi}(X) \mapsto \tilde{\phi}(X) + a + X^{\alpha\beta} a_{\alpha\beta}
\]
Conclusions and discussion

We have obtained the superfield equations in $\mathcal{N} = 2, 4, 8$ extended tensorial superspaces, describing the supermultiplets of the $D = 4$ FMCHS theory with $\mathcal{N}$-extended supersymmetry.
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Extended Supersymmetry in MCHST
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We have elaborated the cases $\mathcal{N} = 2, 4, 8$ since for $n = 4$ these have clear 'lower spin' $D = 4$ counterparts.

The $\mathcal{N} = 2$ supermultiplet is simply given by the complexification of the $\mathcal{N} = 1$ one.
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The $\mathcal{N} = 2$ supermultiplet is simply given by the complexification of the $\mathcal{N} = 1$ one.

For $\mathcal{N} = 4, 8$ the $\mathcal{N}$-extended superfields contain higher components carrying symmetric tensor representations of $GL(4, \mathbb{R})$ that satisfy first order equations in tensorial superspace.
However, this does not imply that we found non-trivial generalizations of the Maxwell, Rarita-Schwinger and linearized conformal gravity equations to tensorial SSP.

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These additional scalar and spinor fields enter the basic superfield under derivatives, so that the theory is invariant under Peccei-Quinn-like symmetries shifting these fields.
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Our superfield equations were found for the case of even $\mathcal{N}$. It would be interesting to consider also the case of odd $\mathcal{N} > 1$ and to look for any specific properties of the $\mathcal{N}$-extended supermultiplets of the MCHS fields with odd $\mathcal{N}$.
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**Our superfield equations were found for the case of even \( N \). It would be interesting to consider also the case of odd \( N > 1 \) and to look for any specific properties of the \( N \)-extended supermultiplets of the MCHS fields with odd \( N \).**

**Generalize the construction to the \( OSp(N|2n) \) supergroup manifolds, which provide the AdS generalization of the tensorial (super)saces.**
Thank you for your attention
App: Quantization of the superparticle in $\Sigma\left(\frac{n(n+1)}{2}|Nn\right)$ with even $N$ and conformal higher spin equations

The various momenta are given by the differential operators

$$P_{\alpha\beta} = -i\partial_{\alpha\beta}, \quad \pi_{\alpha I} = -i\frac{\partial}{\partial\theta_{\alpha I}}, \quad \bar{\eta}_q = \frac{\partial}{\partial\eta_q}$$
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The quantum constraints operators are then

$$\mathcal{D}_{\alpha l} := \frac{\partial}{\partial\theta^{\alpha l}} + i\partial_{\alpha\beta}\theta^{\beta l} - \chi^l\lambda_\alpha, \quad \mathcal{P}_{\alpha\beta} := \partial_{\alpha\beta} - i\lambda_{\alpha}\lambda_\beta$$
App: Quantization of the superparticle in $\sum \binom{n(n+1)}{2} |Nn\rangle$

with even $N$ and conformal higher spin equations

It is convenient to introduce complex Grassmann coordinates, complex Grassmann derivatives and conjugate pairs of fermionic constraints

$$\Theta^{\alpha q} = \frac{1}{2} (\theta^{\alpha q} - i\theta^{\alpha (q+N/2)}) = (\bar{\Theta}_{\bar{q}})^* ,$$

$$\partial_{\alpha q} := \frac{\partial}{\partial \Theta^{\alpha q}} = \frac{\partial}{\partial \theta^{\alpha q}} + i \frac{\partial}{\partial \theta^{\alpha (q+N/2)}} ,$$

$$\nabla_{\alpha q} := D_{\alpha q} + iD_{\alpha (q+N/2)} =: D_{\alpha q} - 2\lambda_\alpha \frac{\partial}{\partial \eta^q} ,$$

$$\bar{\nabla}_{\alpha q} := D_{\alpha q} - iD_{\alpha (q+N/2)} =: \bar{D}_{\alpha q} - 2\lambda_\alpha \eta^q$$
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with even $N$ and conformal higher spin equations

It is sufficient to impose on the WF the fermionic constraints

\[ \nabla_{\alpha q} W := \mathcal{D}_{\alpha q} W - 2\lambda_\alpha \frac{\partial}{\partial \eta^q} W = 0, \]
\[ \bar{\nabla}^p_\beta W := \bar{\mathcal{D}}^q_\alpha W - 2\lambda_\alpha \eta^q W = 0 \]
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The mass–shell–like bosonic constraint will follow as a consistency condition

$$\mathcal{P}_{\alpha \beta} W := (\partial_{\alpha \beta} - i\lambda_\alpha \lambda_\beta) W = 0$$
However, one can easily show that the general solution is expressed through a new complex scalar superfield $\tilde{\phi}(X)$.
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We can begin by decomposing the complex symmetric $GL(4)$ tensor on the $2 \times 2$ blocks, keeping the $GL(2, \mathbb{C})$ manifest symmetry

\[
F_{\alpha\beta} = \begin{pmatrix} F_{AB} & V_{A\dot{B}} \\ V_{B\dot{A}} & F_{\dot{A}\dot{B}} \end{pmatrix}, \quad A, B = 1, 2, \quad \dot{A}, \dot{B} = 1, 2
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\]

The block components which contain the antisymmetric tensors are equivalent to the Maxwell equations:

\[
\partial_{\dot{A}[B} F_{C]D} = 0, \quad \partial_{[A[B} F_{\dot{C}]D} = 0
\]
The components which contain the complex vector $V_{A\dot{B}} = \sigma^a_{A\dot{B}} V_a$

\[ \partial_{A[\dot{B}} V_{\dot{C}]D} = 0, \quad \partial_{B[A} V_{D]\dot{C}} = 0, \]
\[ \partial_{A[B} V_{C]\dot{D}} = 0, \quad \partial_{\dot{A}[\dot{B}} V_{\dot{C}]D} = 0 \]
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\partial_{A[\dot{B}} V_{C]\dot{D}} = 0, \quad \partial_{\dot{A}[\dot{B}} V_{\dot{C}]D} = 0
\]

Solved in the spin-tensor notation

\[
V_{A\dot{B}} = \partial_{A\dot{B}} \tilde{\phi}, \quad \partial_{A[\dot{B}} \partial_{\dot{C}]} \tilde{\phi} = 0
\]
The components which contain vector and antisymmetric tensor

\[ \partial_{\dot{A}\dot{B}} (F_{\dot{C}\dot{D}} - \partial_{\dot{C}\dot{D}} \tilde{\phi}) = 0 , \quad \partial_{A\dot{B}} (\tilde{F}_{\dot{C}\dot{D}} - \partial_{\dot{C}\dot{D}} \tilde{\phi}) = 0 \]
The components which contain vector and antisymmetric tensor

\[ \partial_{A\dot{B}}(F_{CD} - \partial_{CD}\tilde{\phi}) = 0, \quad \partial_{A\dot{B}}(F_{\dot{C}\dot{D}} - \partial_{\dot{C}\dot{D}}\tilde{\phi}) = 0 \]

The only covariant solution of which is given by

\[ F_{CD} = \partial_{CD}\tilde{\phi}, \quad F_{\dot{C}\dot{D}} = \partial_{\dot{C}\dot{D}}\tilde{\phi} \]
App: $\mathcal{N} = 4$

The components which contain vector and antisymmetric tensor

$$\partial_{AB}(F_{CD} - \partial_{CD}\tilde{\phi}) = 0, \quad \partial_{\dot{A}\dot{B}}(F_{\dot{C}\dot{D}} - \partial_{\dot{C}\dot{D}}\tilde{\phi}) = 0$$

The only covariant solution of which is given by

$$F_{CD} = \partial_{CD}\tilde{\phi}, \quad F_{\dot{C}\dot{D}} = \partial_{\dot{C}\dot{D}}\tilde{\phi}$$

One finds that the scalar field $\tilde{\phi}(X)$ satisfies

$$\partial_{A[B}\partial_{C]D}\tilde{\phi} = 0, \quad \partial_{\dot{A}[\dot{B}\partial_{\dot{C]}\dot{D}\tilde{\phi} = 0$$