Neutrino Mixing and Neutrino Telescopes

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Astrophysical Neutrinos:

- General Properties of Neutrino Mixing Probabilities
- Expansion of Probabilities up to second Order
- Dependence of Flux Ratios on "impure" initial Flux Compositions

W.R., JCAP **0701**, 029 (2007);

S. Pakvasa, W.R., T.J. Weiler, in preparation

Neutrino Mixing

"Best-fit matrix" with vanishing U_{e3} and maximal θ_{23}:

$$|U| = |U_\ell^\dagger \, U_\nu| = \begin{pmatrix} 0.83 & 0.56 & 0 \\ 0.39 & 0.59 & 1/\sqrt{2} \\ 0.39 & 0.59 & 1/\sqrt{2} \end{pmatrix}$$

"Tri–bimaximal Mixing" $\sin^2\theta_{12} = 1/3$:

$$U = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

$$\Leftrightarrow m_\nu = \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}(A+B+D) & \frac{1}{2}(A+B-D) \\ \cdot & \cdot & \frac{1}{2}(A+B+D) \end{pmatrix}$$

Neutrino Mixing and Neutrino Telescopes

Measure Flux Ratios at Neutrino Telescopes

initial flux composition of neutrinos:

$$\begin{pmatrix} \Phi_e^0 \\ \Phi_\mu^0 \\ \Phi_\tau^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

measured is

$$\begin{pmatrix} \Phi_e \\ \Phi_\mu \\ \Phi_\tau \end{pmatrix} = \begin{pmatrix} P_{ee} & P_{e\mu} & P_{e\tau} \\ P_{\mu e} & P_{\mu\mu} & P_{\mu\tau} \\ P_{\tau e} & P_{\tau\mu} & P_{\tau\tau} \end{pmatrix} \begin{pmatrix} \Phi_e^0 \\ \Phi_\mu^0 \\ \Phi_\tau^0 \end{pmatrix}$$

with neutrino mixing probability

$$P_{\alpha\beta} = \sum_i |U_{\alpha i}|^2 \, |U_{\beta i}|^2$$

Properties of Mixing Probabilities I

- tri-bimaximal mixing:

$$P_{\text{TBM}} = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\ \cdot & \frac{7}{18} & \frac{7}{18} \\ \cdot & \cdot & \frac{7}{18} \end{pmatrix}$$

- allowed ranges:

$$P = \begin{cases} \begin{pmatrix} 0.51 \div 0.62 & 0.13 \div 0.31 & 0.13 \div 0.32 \\ \cdot & 0.34 \div 0.50 & 0.33 \div 0.40 \\ \cdot & \cdot & 0.33 \div 0.49 \end{pmatrix} & \text{at } 2\sigma \\ \begin{pmatrix} 0.48 \div 0.64 & 0.11 \div 0.34 & 0.11 \div 0.35 \\ \cdot & 0.33 \div 0.54 & 0.30 \div 0.41 \\ \cdot & \cdot & 0.33 \div 0.52 \end{pmatrix} & \text{at } 3\sigma \end{cases}$$

Astrophysical Neutrinos mix!!

Properties of Mixing Probabilities II

- depends on $\sin\theta_{12}$, $\sin\theta_{23}$, $\sin\theta_{13}$ and $\mathrm{Re}\{U_{e3}\} = |U_{e3}|\cos\delta$
- $P_{\alpha\beta} = P_{\beta\alpha} \Rightarrow$ six independent probabilities
- $\sum_\alpha P_{\alpha\beta} = \sum_\beta P_{\alpha\beta} = 1$
- if you know $P_{e\mu}$ and $P_{\mu\mu}$ (and θ_{23}):

$$P_{e\tau} = P_{e\mu}(\theta_{23} \to \theta_{23} + \pi/2)$$

$$P_{\tau\tau} = P_{\mu\mu}(\theta_{23} \to \theta_{23} + \pi/2)$$

$$P_{ee} = 1 - P_{e\mu} - P_{e\tau}$$

$$P_{\mu\tau} = 1 - P_{e\mu} - P_{\mu\mu}$$

$\Rightarrow$ only two are independent

The probabilities

$$P_{e\mu} = 2\, c_{13}^2 \left(c_{12}^2\, s_{12}^2\, c_{23}^2 + \left(c_{12}^4 + s_{12}^2 \right) s_{13}^2\, s_{23}^2 \right.$$

$$\left. + c_{12}\, s_{12}\, c_{23}\, s_{23}\, c_\delta\, (c_{12} - s_{12})(c_{12} + s_{12})\, s_{13} \right)$$

$$P_{\mu\mu} = 1 - 2\, c_{12}^4\, c_{23}^2\, s_{23}^2\, s_{13}^2$$

$$+2\left(\left(s_{12}^2\left(\left(s_{13}^4 + \left(4\, c_\delta^2 - 1\right) s_{13}^2 + 1\right) s_{23}^2 - 1\right) - c_{13}^2\, s_{23}^2\right) c_{23}^2\right.$$

$$+ s_{13}^2\, s_{23}^2 \left(c_{13}^2\, s_{12}^2 - \left(c_{13}^2 + s_{12}^2\right) s_{23}^2\right)\right) c_{12}^2$$

$$+ s_{23}\left(-2\left(c_{23}^2\, c_{13}^4 + \left(c_{13}^2 + c_{23}^2\, s_{12}^2\right) s_{13}^2\right) s_{23}\, s_{12}^2\right.$$

$$-2\, c_{12}\, s_{12}\, c_{23}\, c_\delta\, (c_{12} - s_{12})(c_{12} + s_{12})$$

$$\left.\left(c_{13}^2 + \left(s_{13}^2 + 1\right)(c_{23} - s_{23})(c_{23} + s_{23})\right) s_{13}\right)$$

Better to expand them...

Expansion of the Probabilities I

We know of two small parameters:

$$|U_{e3}| \text{ and } \epsilon = \frac{\pi}{4} - \theta_{23}$$

$\Rightarrow$ expand and truncate after the quadratic terms:

$$P_{ee} \simeq (1 - 2\, c_{12}^2\, s_{12}^2)(1 - 2\,|U_{e3}|^2)$$

$$P_{e\mu} \simeq c_{12}^2\, s_{12}^2 + \Delta + (1 - 2\, c_{12}^2\, s_{12}^2)\,|U_{e3}|^2$$

$$P_{e\tau} \simeq c_{12}^2\, s_{12}^2 - \Delta + (1 - 2\, c_{12}^2\, s_{12}^2)\,|U_{e3}|^2$$

$$P_{\mu\mu} \simeq \frac{1}{2}\,(1 - c_{12}^2\, s_{12}^2) - \Delta + \frac{1}{2}\,\overline{\Delta}^2 - \frac{1}{2}\,(1 - 2\, c_{12}^2\, s_{12}^2)\,|U_{e3}|^2$$

$$P_{\mu\tau} \simeq \frac{1}{2}\,(1 - c_{12}^2\, s_{12}^2) - \frac{1}{2}\,\overline{\Delta}^2 - \frac{1}{2}\,(1 - 2\, c_{12}^2\, s_{12}^2)\,|U_{e3}|^2$$

$$P_{\tau\tau} \simeq \frac{1}{2}\,(1 - c_{12}^2\, s_{12}^2) + \Delta + \frac{1}{2}\,\overline{\Delta}^2 - \frac{1}{2}\,(1 - 2\, c_{12}^2\, s_{12}^2)\,|U_{e3}|^2$$

Expansion of the Probabilities II

$$P_{\mu\mu} \simeq \frac{1}{2}\left(1 - c_{12}^2\, s_{12}^2\right) - \Delta + \frac{1}{2}\,\overline{\Delta}^2 - \frac{1}{2}\,(1 - 2\,c_{12}^2\, s_{12}^2)\,|U_{e3}|^2$$

universal first and second order terms ($\epsilon = \frac{\pi}{4} - \theta_{23}$):

$$\Delta \equiv \frac{1}{4}\,\sin 4\theta_{12}\,\cos\delta\,|U_{e3}| + \frac{1}{2}\,\sin^2 2\theta_{12}\,\epsilon$$

$$\overline{\Delta}^2 \equiv \sin^2 2\theta_{12}\,\cos^2\delta\,|U_{e3}|^2 + 4\,(1 - \cos^2\theta_{12}\,\sin^2\theta_{12})\,\epsilon^2 - \sin 4\theta_{12}\,\cos\delta\,|U_{e3}|\,\epsilon$$

- terms proportional to $|U_{e3}|^2$ negligible (and they cancel for 1 : 2 : 0)
- $\overline{\Delta}^2$ only in $\mu\tau$ sector
- always $\overline{\Delta}^2 \geq 0$

Properties of Δ and $\overline{\Delta}^2$

$$1\sigma: \quad -0.036 \le \Delta \le 0.057 \quad , \quad 3\sigma: \quad -0.097 \le \Delta \le 0.113$$

$$1\sigma: \quad \overline{\Delta}^2 \le 0.042 \quad , \quad 3\sigma: \quad \overline{\Delta}^2 \le 0.170$$

Second order term can exceed first order term!

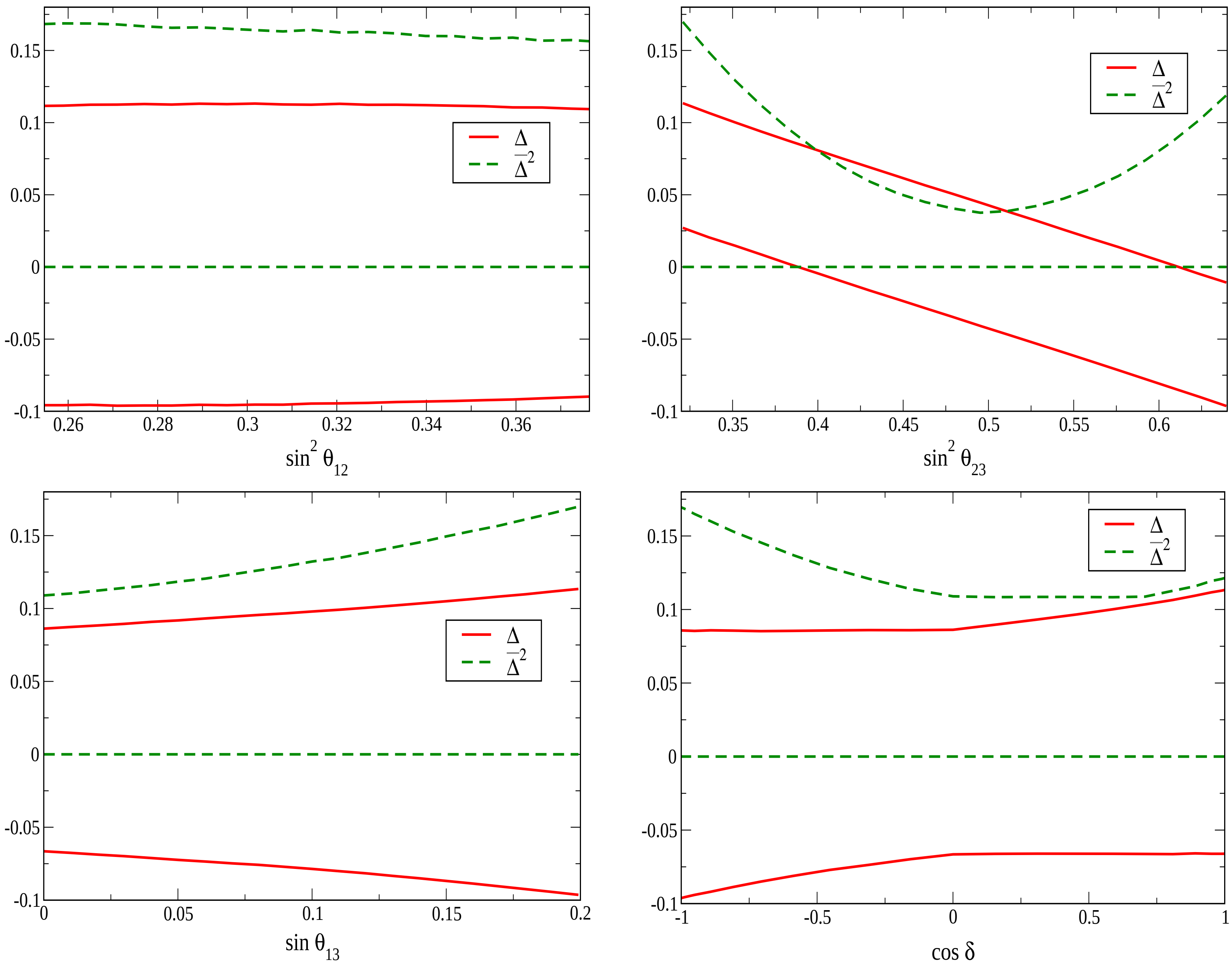
$$\text{if } \Delta = 0 : \begin{cases} \overline{\Delta}^2 \le 0.016 & (1\sigma) \\ \overline{\Delta}^2 \le 0.119 & (3\sigma) \end{cases}$$

if $\sin^2\theta_{12} = \frac{1}{3}$:

$$\Delta = \frac{1}{9}\left(\sqrt{2}\,\cos\delta\,|U_{e3}| + 4\,\epsilon\right)$$

$$\overline{\Delta}^2 = \frac{4}{9}\left(2\cos^2\delta\,|U_{e3}|^2 + 7\,\epsilon^2 - \sqrt{2}\,\cos\delta\,|U_{e3}|\,\epsilon\right)$$

$\Rightarrow$ Depends most strongly on θ_{23}

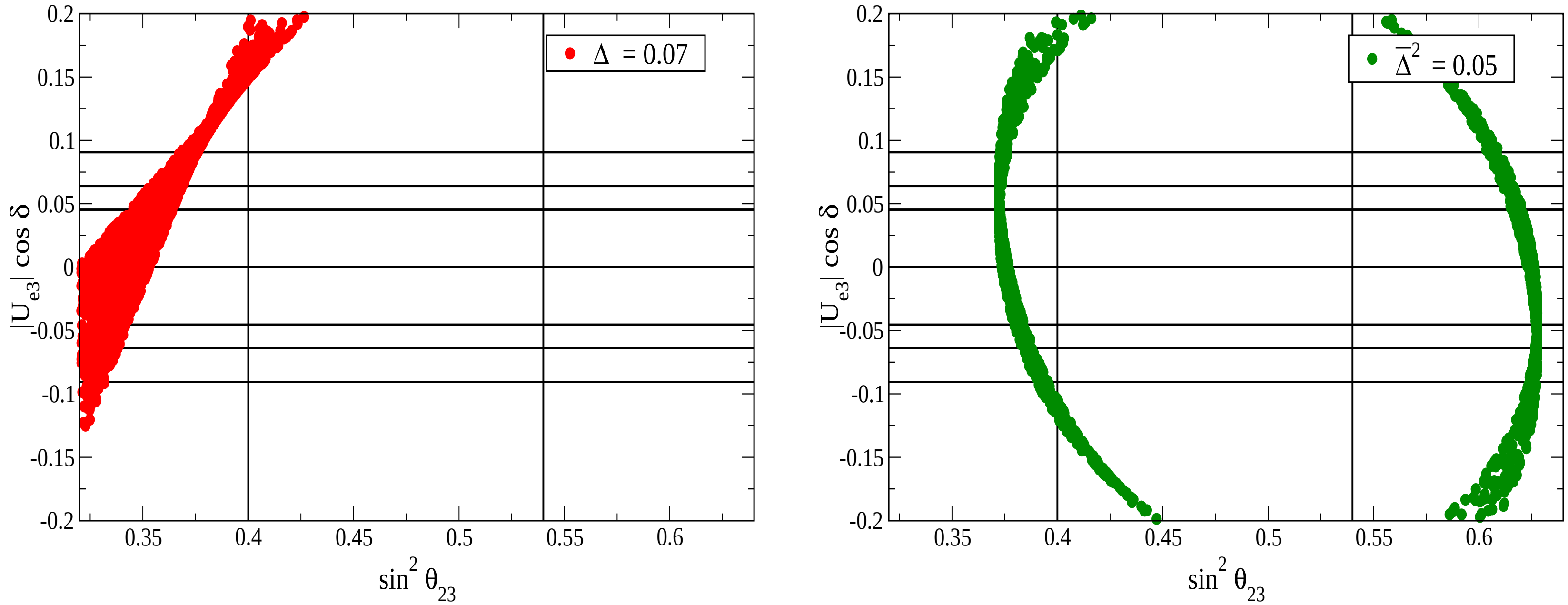


0.15
0.1
0.05
0
-0.05
-0.1
0.26
0.28
0.3
0.32
0.34
0.36
$\sin^2\theta_{12}$
0.35
0.4
0.45
0.5
0.55
0.6
$\sin^2\theta_{23}$
0
0.05
0.1
0.15
0.2
$\sin\theta_{13}$
-1
-0.5
0
0.5
1
$\cos\delta$
Δ
$\bar{\Delta}^2$

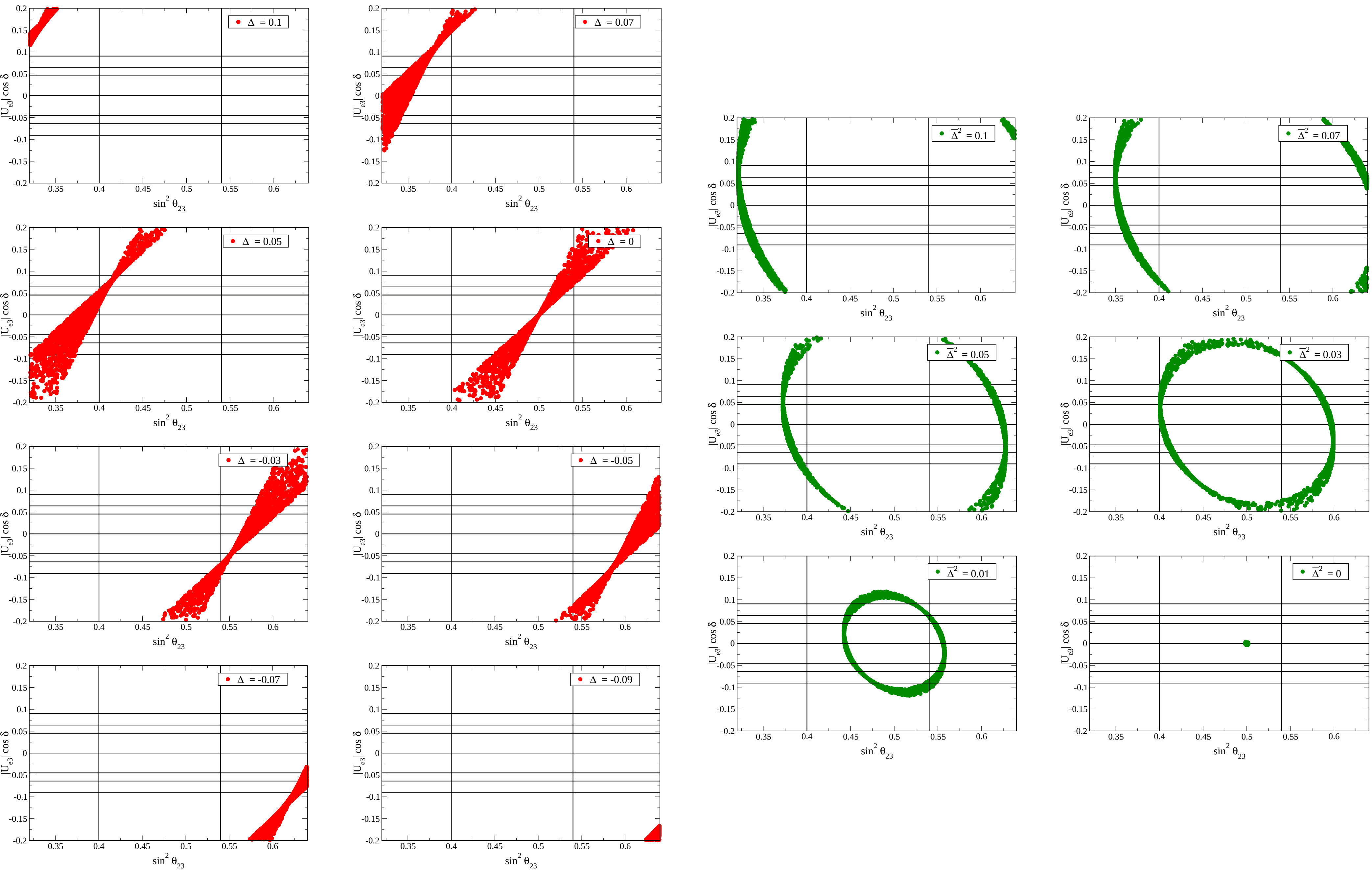
Properties of Δ and $\overline{\Delta}^2$

Measure flux ratio $\Rightarrow$ measure Δ and/or $\overline{\Delta}^2$

are functions of $|U_{e3}|\cos\delta$ and $\sin^2\theta_{23}$



- $\Delta = 0$ also for $|U_{e3}|\cos\delta = \left(\sin^2\theta_{23} - \frac{1}{2}\right)\tan 2\theta_{12}$
- $\overline{\Delta}^2$ describes ellipses, zero only for $U_{e3} = \theta_{23} - \pi/4 = 0$


Δ = 0.1
Δ = 0.07
Δ = 0.05
Δ = 0
Δ = -0.03
Δ = -0.05
Δ = -0.07
Δ = -0.09
$\bar{\Delta}^2$ = 0.1
$\bar{\Delta}^2$ = 0.07
$\bar{\Delta}^2$ = 0.05
$\bar{\Delta}^2$ = 0.03
$\bar{\Delta}^2$ = 0.01
$\bar{\Delta}^2$ = 0
$\sin^2 \theta_{23}$
$|U_{e3}| \cos \delta$

Simplified Case

Modified Bjorken-Harrison-Scott (Phys. Rev. D **74**, 073012 (2006)) parametrization:

$$U = \begin{pmatrix} c\,D & s & U_{e3} \\ -\frac{s}{\sqrt{2}}\,D - \frac{1}{\sqrt{2}}\frac{1}{c}\,U_{e3}^* & \frac{c}{\sqrt{2}} & \sqrt{\frac{1}{2}}\,D - \frac{s}{c\,\sqrt{2}}\,U_{e3} \\ -\frac{s}{\sqrt{2}}\,D + \frac{1}{\sqrt{2}}\,\frac{1}{c}\,U_{e3}^* & \frac{c}{\sqrt{2}} & -\sqrt{\frac{1}{2}}\,D - \frac{s}{c\,\sqrt{2}}\,U_{e3} \end{pmatrix}$$

$$\text{with } D = \sqrt{1 - \tfrac{1}{c^2}\,|U_{e3}|^2}$$

(originally: $c = \sqrt{2/3}$, lead to too large $\sin^2\theta_{12} = \frac{1}{3}(1 + |U_{e3}|^2)$)

$$\Delta \simeq \epsilon\,\cos^2\theta_{12} \quad \text{and} \quad \overline{\Delta}^2 \simeq 4\,\epsilon^2$$

now proportional to each other

Flux Ratios and Sources

Initial flux composition $\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0$

- pion sources: $1 : 2 : 0$
- muon-damped: $0 : 1 : 0$
- neutron beam: $1 : 0 : 0$

Example: $1 : 2 : 0$ goes to

$$\Phi_e : \Phi_\mu : \Phi_\tau = 1 + 2\,\Delta : 1 - \Delta + \overline{\Delta}^2 : 1 - \Delta - \overline{\Delta}^2 \neq 1 : 1 : 1$$

Therefore always more ν_μ than ν_τ:

$$\frac{\Phi_\mu}{\Phi_\tau} \simeq 1 + 2\,\overline{\Delta}^2 \geq 1$$

Many papers on how to test neutrino parameters:

Bhattacharjee, Gupta; Serpico, Kachelriess; Xing; Kachelriess, Tomas; Meloni, Ohlsson; Blum, Nir, Waxman; Xing, Zhou; W.R.; Winter; Choubey, Awasthi;...

Example

$$\frac{\Phi_\mu}{\Phi_{\rm tot}} \simeq \begin{cases} \frac{1}{3}\left(1 - \Delta + \overline{\Delta}^2\right) & \text{pion} \\ \frac{1}{2}(1 - c_{12}^2\, s_{12}^2) - \Delta + \frac{1}{2}\,\overline{\Delta}^2 \longrightarrow \frac{7}{18} - \Delta + \frac{1}{2}\,\overline{\Delta}^2 & \text{muon-damped} \\ c_{12}^2\, s_{12}^2 + \Delta \longrightarrow \frac{2}{9} + \Delta & \text{neutron} \end{cases}$$

$\Rightarrow$ Pion sources are less sensitive to deviations from zero U_{e3} and maximal θ_{23}

$\Rightarrow$ muon-damped and neutron sources depend at first order on θ_{12}

Flux Ratios and Sources

Realistically, one should use:

- pion sources: $1 : 2\,(1 - \zeta) : 0$
- muon-damped: $\eta : 1 : 0$
- neutron beam: $1 : \eta : 0$

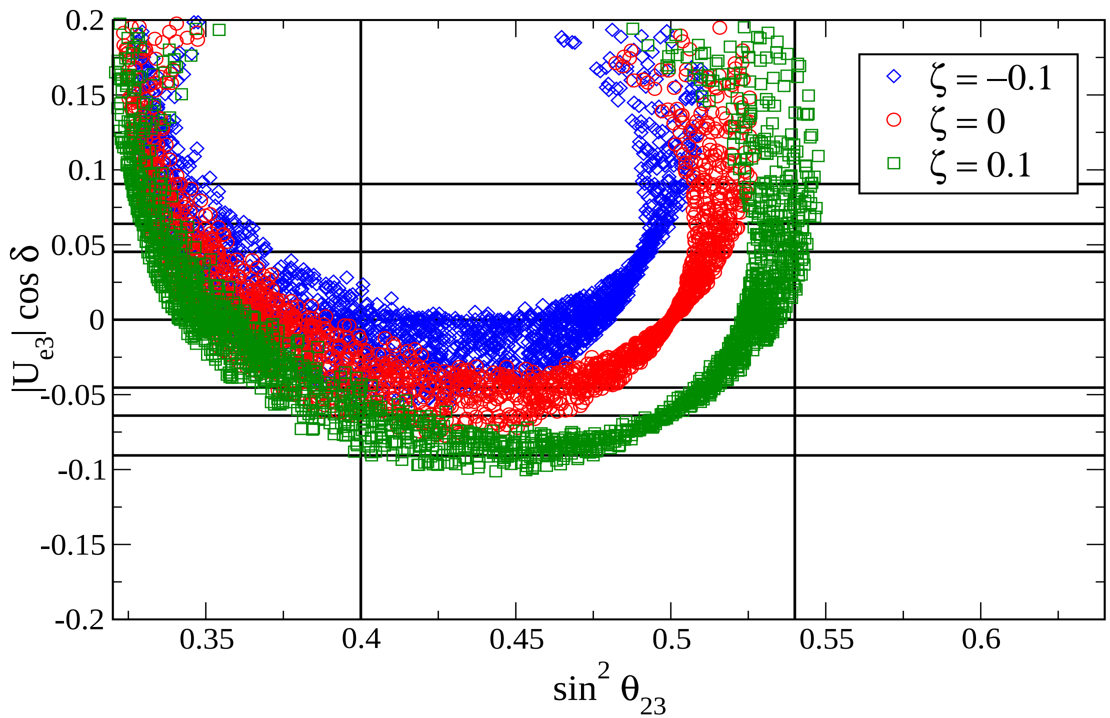
Pakvasa, W.R., Weiler, in preparation

Reasons are

- polarization effect in pion decays, different energy spectra for different flavors, etc., results in $1 : 1.86 : 0$ (Lipari, Lusignoli, Meloni, Phys. Rev. D **75**, 123005 (2007))
- $0 : 1 : 0$ from $1 : 2 : 0$ due to insufficient decay of muons at higher energy $\leftrightarrow$ might not be perfect (Kashti, Waxman, Phys. Rev. Lett. **95**, 181101 (2005); Muecke, Protheroe, Stanev, Astropart. Phys. **18**, 593 (2003))

Effect of Impure initial Flux Compositions

$$1 : 2\,(1 - \zeta) : 0$$

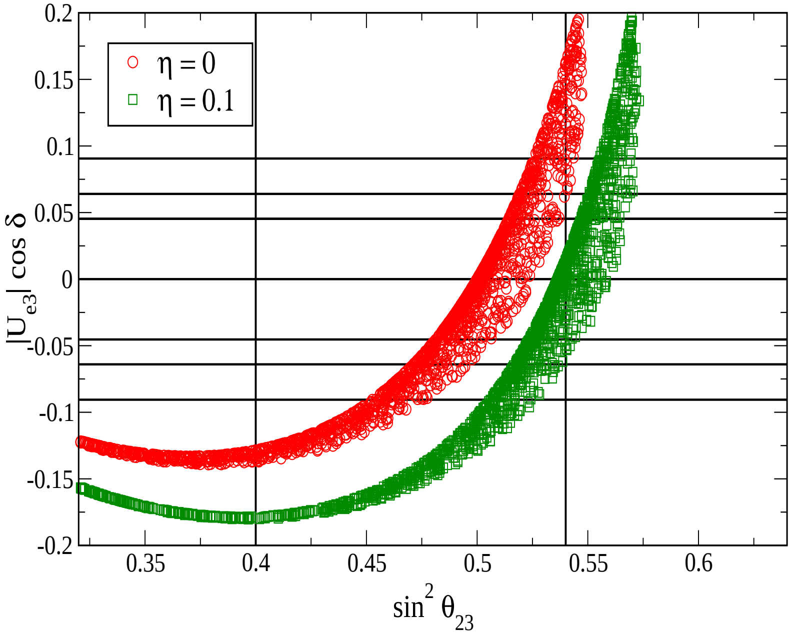


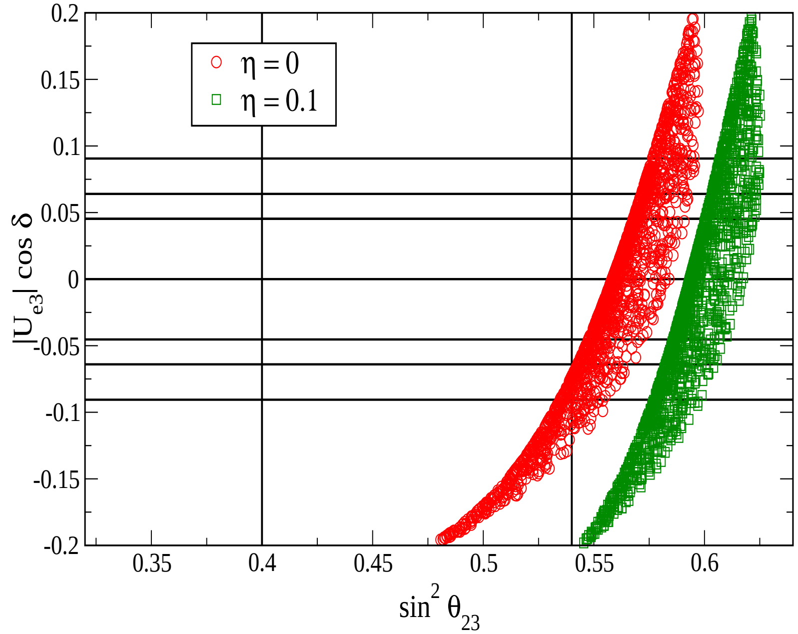
$$\Phi_\mu / \Phi_{\rm tot} = 0.33$$

$$\sin^2\theta_{23} = \tfrac{1}{2} \text{ and } \zeta = 0 \Rightarrow |U_{e3}|\,\cos\delta = 0$$
$$\sin^2\theta_{23} = \tfrac{1}{2} \text{ and } \zeta = 0.1 \Rightarrow |U_{e3}|\,\cos\delta = 0.06$$

Effect of Impure initial Flux Compositions II

$\eta : 1 : 0$



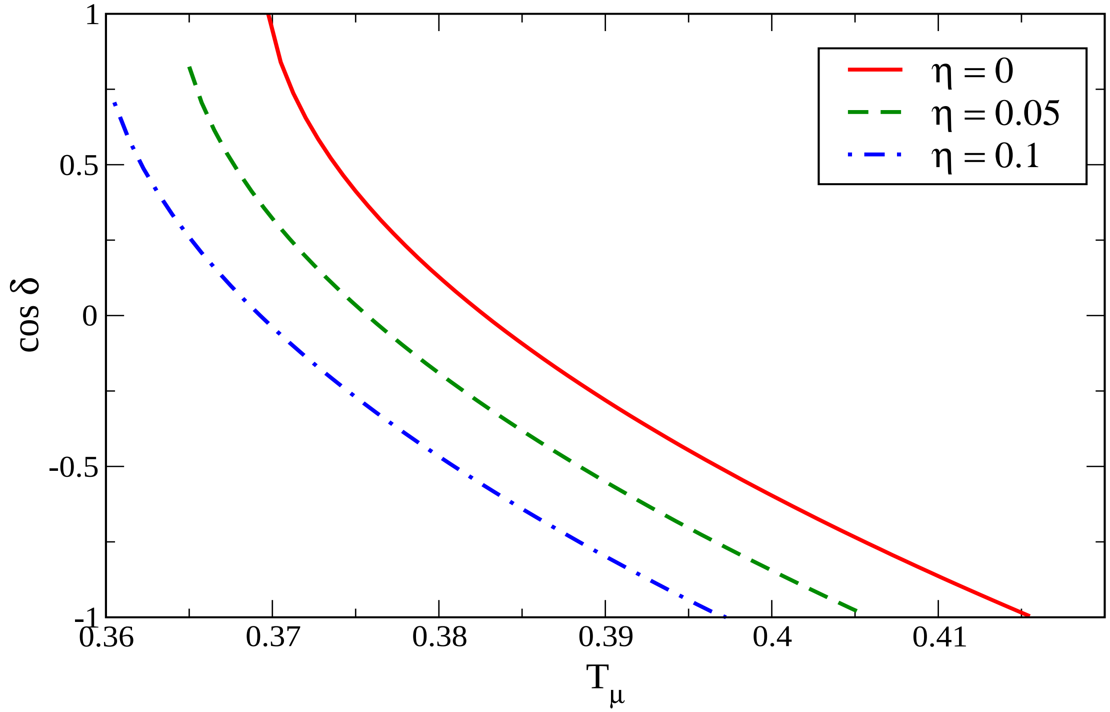


$\Phi_\mu/\Phi_{\rm tot} = 7/18$

$\Phi_\mu/\Phi_{\rm tot} = 0.42$

Effect of Impure initial Flux Compositions IIa

$\eta : 1 : 0$ and CP violation/conservation



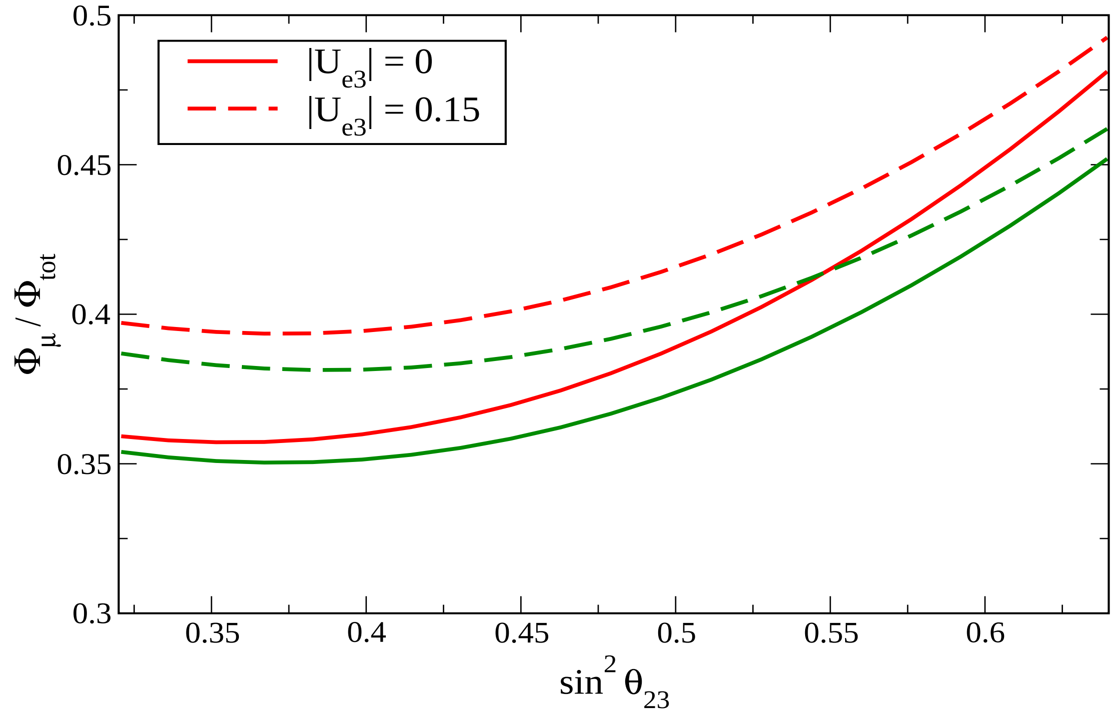
Extracted value of $\cos\delta$ from an exact measurement of $\Phi_\mu/\Phi_{\rm tot}$

$$\theta_{23} = \pi/4,\ \sin^2\theta_{12} = \frac{1}{3} \text{ and } |U_{e3}| = 0.15$$

as in Blum, Nir, Waxman, arXiv:0706.2070 [hep-ph]

Effect of Impure initial Flux Compositions IIb

$\eta : 1 : 0$ and the value of $|U_{e3}|$



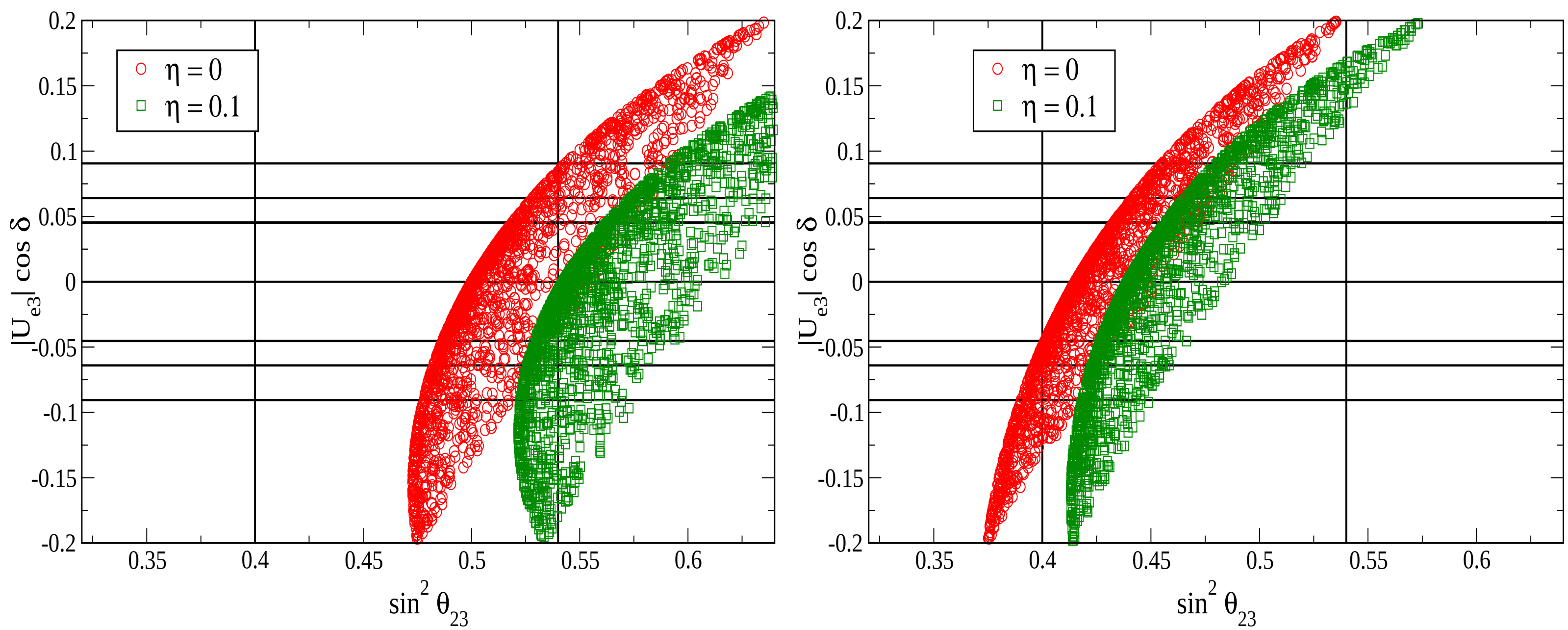
Dependence on $\sin^2\theta_{23}$ of $\Phi_\mu/\Phi_{\rm tot}$ for $\delta = \pi$

plotted for $0 : 1 : 0$ and for $0.1 : 1 : 0$

as in `Serpico, Phys. Rev. D` **`73`**`, 047301 (2006)`

Effect of Impure initial Flux Compositions III

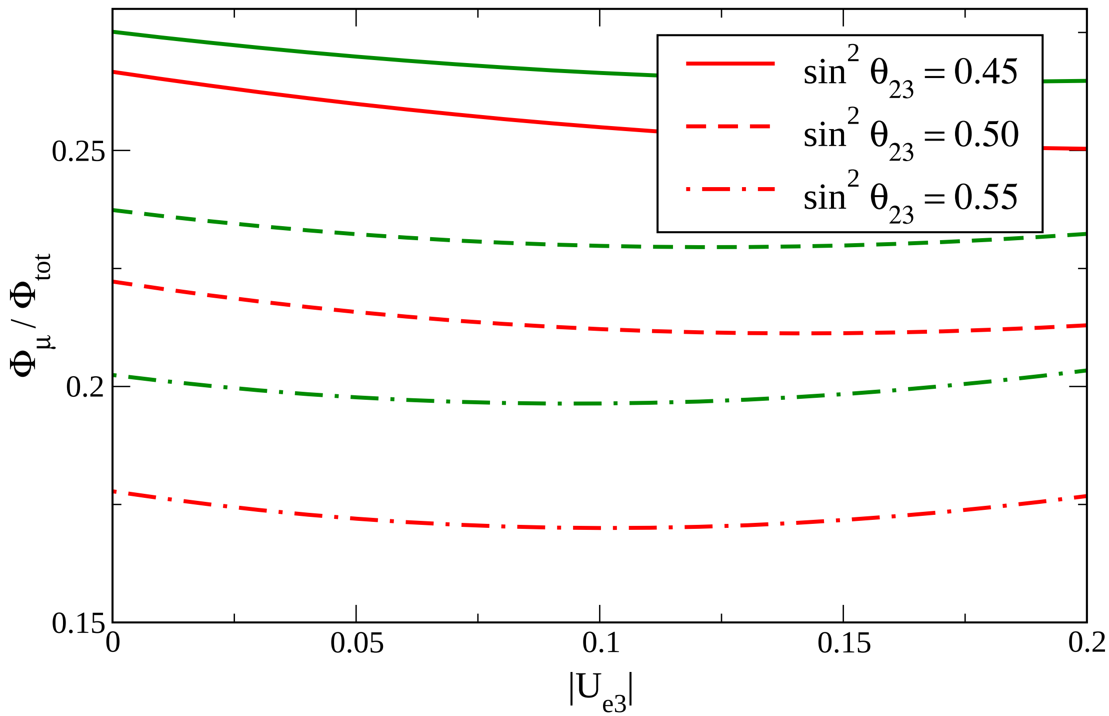
$1 : \eta : 0$



$\Phi_\mu/\Phi_{\rm tot} = 2/9$ $\Phi_\mu/\Phi_{\rm tot} = 0.26$

Effect of Impure initial Flux Compositions IIIa

$\eta : 1 : 0$ and the octant of θ_{23}

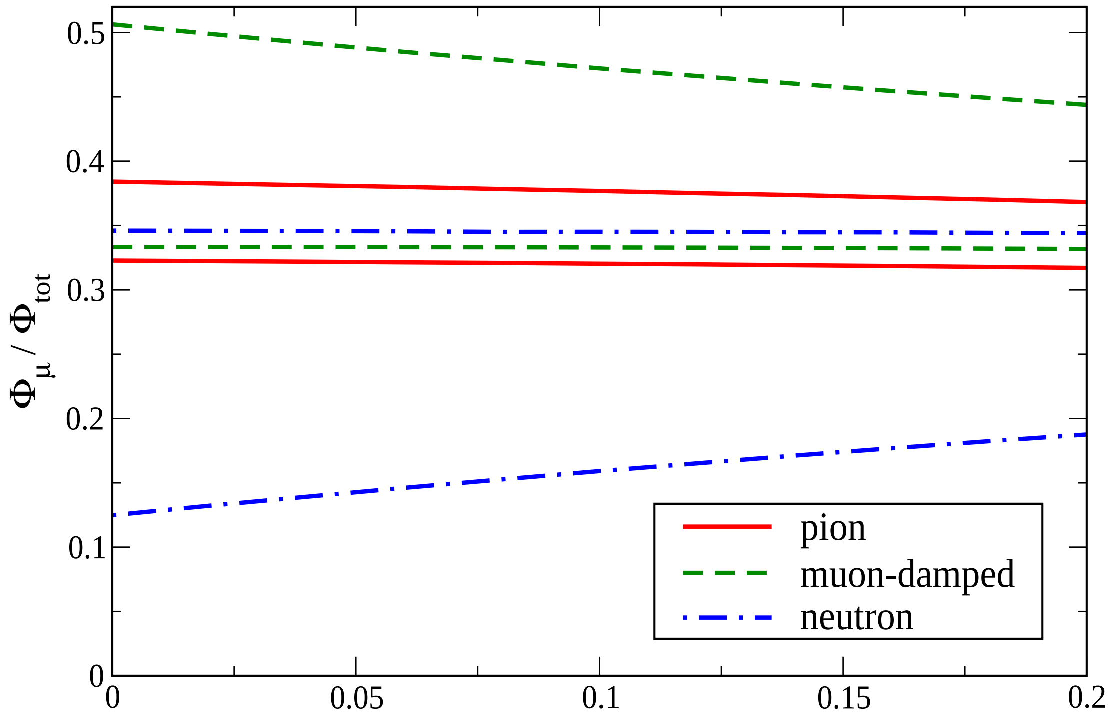


Dependence on θ_{13} of $\Phi_\mu/\Phi_{\text{tot}}$ for $\delta = \pi$

Plotted are $0:1:0$ and $0.1:1:0$

as in `Kachelriess, Serpico, Phys. Rev. Lett.` **`94`**`, 211102 (2005)`

Range of Ratios



$\Phi_\mu/\Phi_{\rm tot}$ for

$$1 : 2\,(1 - \zeta) : 0$$

$$\eta : 1 : 0$$

$$1 : \eta : 0$$

Summary

Flux Ratios can be described by universal first and second order corrections Δ and $\overline{\Delta}^2$

- $\overline{\Delta}^2 \geq 0$ can be larger than $|\Delta|$
- has to be included in analytical studies
- $\Phi_e : \Phi_\mu : \Phi_\tau = 1 + 2\,\Delta : 1 - \Delta + \overline{\Delta}^2 : 1 - \Delta - \overline{\Delta}^2$

$$\begin{pmatrix} 1 \\ 2\,(1-\zeta) \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} \eta \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ \eta \\ 0 \end{pmatrix}$$

$$\frac{\Phi_\mu}{\Phi_{\rm tot}} \simeq \begin{cases} \frac{1}{3}\left(1 - \Delta + \overline{\Delta}^2 - \frac{1}{9}\,\zeta\right) & \text{pion} \\ \frac{7}{18} - \Delta + \frac{1}{2}\,\overline{\Delta}^2 - \frac{1}{6}\,\eta & \text{muon-damped} \\ \frac{2}{9} + \Delta + \frac{1}{6}\,\eta & \text{neutron} \end{cases}$$

⇒ Pion sources are less sensitive to deviations

⇒ muon-damped and neutron sources depend at first order on θ_{12}

⇒ can affect statements about

- CP violation/conservation
- value of $|U_{e3}|$
- value and octant of θ_{23}

Extras

What is μ–τ symmetry?

simple exchange (Z_2 or S_2) symmetry

$$P_{\mu\tau} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ gives } P_{\mu\tau}\,\nu = P_{\mu\tau} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \nu_e \\ \nu_\tau \\ \nu_\mu \end{pmatrix}$$

apply also to mass term $\overline{\nu}\, m_\nu\, \nu^c$

$$P_{\mu\tau} \begin{pmatrix} A & B & D \\ B & E & F \\ D & F & G \end{pmatrix} (P_{\mu\tau}^*)^{-1} = \begin{pmatrix} A & D & B \\ D & G & F \\ B & F & E \end{pmatrix}$$

$$\Rightarrow \text{is symmetry if } \left\{ \begin{array}{l} D = B \\ G = E \end{array} \right. \Rightarrow m_\nu = \begin{pmatrix} A & B & B \\ B & D & E \\ B & E & D \end{pmatrix}$$

What is μ–τ symmetry?

$$m_\nu = \begin{pmatrix} A & B & B \\ B & D & E \\ B & E & D \end{pmatrix} \text{ has eigenvector } \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ to eigenvalue } D - E$$

$$\Rightarrow U_{e3} = 0 \text{ and } \theta_{23} = \pi/4 \text{ !!!!}$$

- 6 free parameters: $m_{1,2,3}$, θ_{12} and Majorana phases
- no Dirac phase δ

Problems

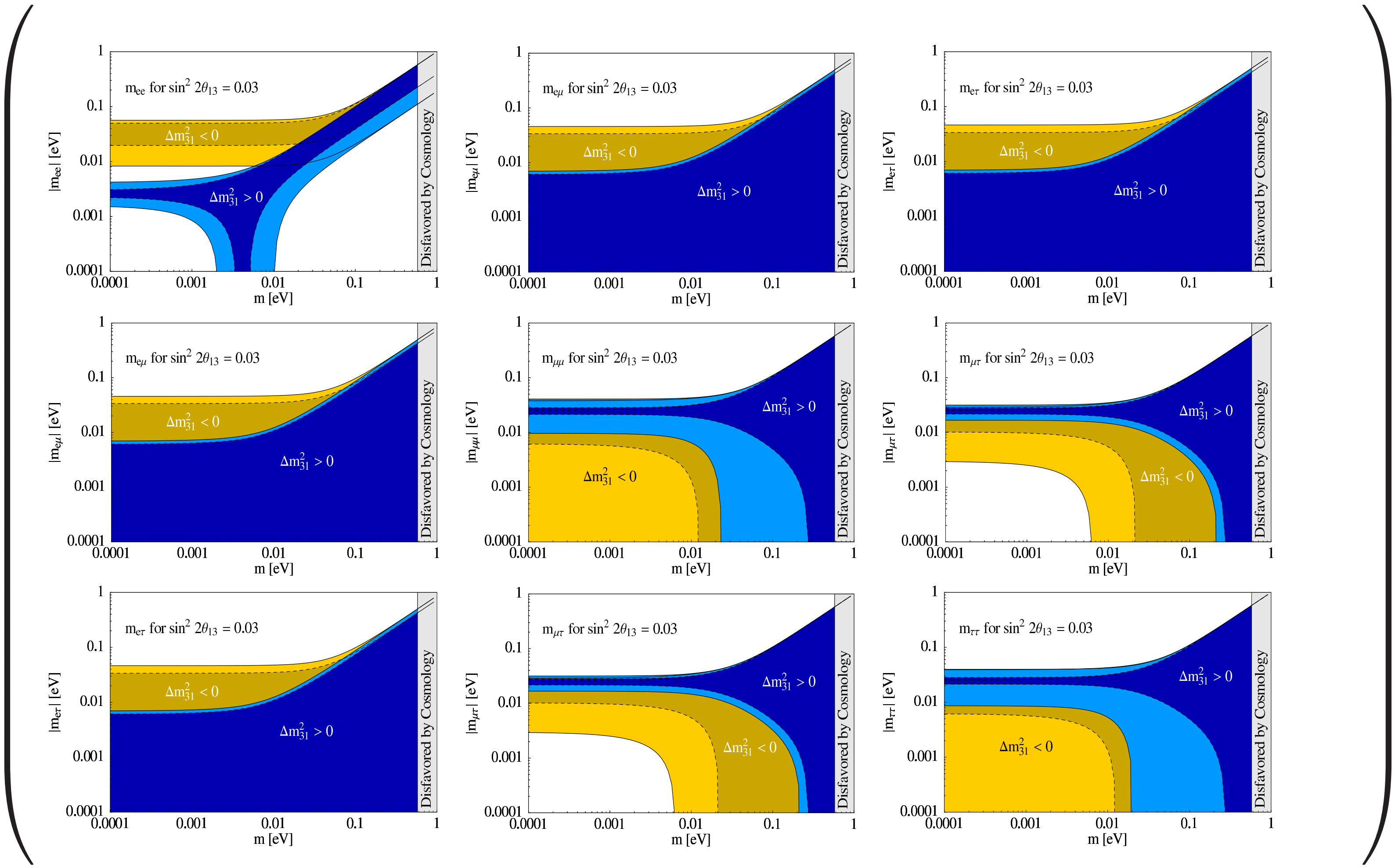
$$m_\nu = \begin{pmatrix} A & B & B \\ B & D & E \\ B & E & D \end{pmatrix} \text{ has eigenvector } \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ to eigenvalue } D - E$$

- no reason that eigenvalue to eigenvector is correct one
- Why not charged leptons? $U = U_\ell^\dagger \, U_\nu$
  - if in symmetry basis charged leptons diagonal: $m_\mu = m_\tau$
  - if in symmetry basis charged leptons NOT diagonal: $\theta_{13} = \theta_{23} = 0$, $\theta_{12} \neq 0$

$*$ more than one Higgs doublet (`Mohapatra; Grimus, Lavoura`)

$*$ or (tuned!) quasi-degenerate neutrinos with equal CP parities and softly broken μ–τ symmetry (`Joshipura; Haba, W.R.`)

$$(m_\nu)_{\alpha\beta} =$$



Two Popular Cases

(i) "Tri–bimaximal Mixing" $\sin^2\theta_{12} = 1/3$:

$$= \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} \Leftrightarrow m_\nu = \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}\,(A+B+D) & \frac{1}{2}\,(A+B-D) \\ \cdot & \cdot & \frac{1}{2}\,(A+B+D) \end{pmatrix}$$

(ii) "Bimaximal Mixing" $\sin^2\theta_{12} = 1/2$:

$$U = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\sqrt{\frac{1}{2}} \\ -\frac{1}{2} & \frac{1}{2} & \sqrt{\frac{1}{2}} \end{pmatrix} \Leftrightarrow m_\nu = \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}\,(A+D) & \frac{1}{2}\,(A-D) \\ \cdot & \cdot & \frac{1}{2}\,(A+D) \end{pmatrix}$$

Note: mixing angles independent of mass matrix entries

Usually symmetries based on finite groups $A_4, D_5, S_3, \ldots$, typically with Z_2

Babu, Ma, Valle; Altarelli, Feruglio; Hagedorn; Frampton, Kephart; King, Ross; Ma; Ma; Ma; Ma; Ma; Ma; Ma; Ma; Ma; Ma;...